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Structural Implications of using Cairo Tiling and Hexagons in Gridshells

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Summary: This paper explores using Cairo tiling (pentagonal) and hexagonal patterns for the design of gridshells. The structural performance of these grids is measured by calculating the failure load using the finite element method. It is found that pentagonal grids are comparable in capacity to quadrilaterals. Hexagonal grids have a smaller capacity but are as structurally efficient as the pentagonal grids. Both the pentagonal and hexagonal grids offer new aesthetical possibilities.

Keywords: *gridshell, lattice shell, Cairo tiling, structural analysis*

1. INTRODUCTION

A gridshell is constructed of a grid or lattice of linear structural members. Timber gridshells usually have the members crossing above and below each other at the nodes while steel gridshells have individual members bolted or welded to the nodes. The openings of the grid are often covered with glass panels to allow for daylight, but can be clad with other materials such as wood or EPTFE. Gridshells are often used to enclose existing spaces, as was done for the British Museum Great Court in London and the Smithsonian American Art Museum and National Portrait Gallery in Washington DC, but they can also serve as stand-alone structures such as the Weald and Downland Museum [1].

The majority of built gridshells use quadrilateral or triangular grids. In this paper, we investigate the structural efficiency of gridshells with Cairo-tiled (pentagonal) and hexagonal grids. This study was inspired by our work with engineers Atelier One on the pentagonal gridshell design for the Inhotim sculpture park in Minas Gerais for the client Bernardo Paz, and the architect Pedro Doyle.

We also drew inspiration from cellular structures seen in nature such as honeycombs or dragonfly wings. Nature is often a source of inspiration in architectural and structural design. Early examples of mimicking nature both aesthetically and technically include Rene Binet's Porte Monumentale for the 1900 Paris Expo, La which was inspired by Earnst Haeckel's detailed drawings of microscopic sea creatures, radiolaria [2]. Frei Otto also used this tradition of nature to inspire aesthetical and technical forms such as his spider web like tensile structure, the Munich Olympic Stadium.

The lattice patterns seen in nature are more irregular and randomized, offering more structural redundancy. The goal of this paper is to provide the basis for future investigation of randomized grids by first assessing the structural capacity of the regular pentagonal and hexagonal gridshells.

Shell structures are 3D structures that are thin in one dimension. They have the ability to span large areas with a minimum amount of material. Shells take loads effectively because of their form; they have additional stiffness due to their curvature. The detailed analysis of a shell structure is difficult because it is highly sensitive to its curvature and any imperfections; given the same material and thickness, if the curvature is changed, the shell can have a different failure load and mode [3].

Gridshells are a subsection of shells because they also use curvature to increases their stiffness and take loading while maintaining relatively thin members. However, gridshells usually rely more upon bending than do continuous shells.

In this paper, we study the structural capacity of a spherical cap gridshell. The dominant failure mode for a continuous spherical cap is global buckling. Timoshenko derived the analytical buckling load for a shell of thickness t, radius R, Young's Modulus E, and Poisson's ratio ν, presented in eq. 1. Timoshenko's derivation assumes small displacements and a symmetric buckling mode about the diameter [4].

$$q_{cr} = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{R}\right)^2 \tag{1}$$

As the shell becomes thinner, the buckling load decreases to the power two. As the radius increases the buckling load decreases quadratically.

The closed-form solution of the buckling load of a continuous shell has been extrapolated to that of a gridshell through the use of the equivalent continuum [5]. The equivalent continuum is often defined by an equivalent thickness that describes the transition from the discrete gridshell to the continuous. The difficulty in defining the equivalent thickness is in understanding the interplay between the in-plane membrane response and the out-of-plane bending response because the bending and axial stiffness include shell thickness raised to a different power. As a result, the equivalent thickness can be defined in many ways [5,6,7].

It should be emphasised that even if a shell takes loads primarily by membrane action, bending stiffness is still required to resist buckling and therefore a gridshell should never be made with pin-jointed nodes.

The parameters that govern the structural behaviour of the gridshell are the grid spacing, the joint rigidity, the grid topology, the member section, the global curvature and boundary conditions [8]. In this study, we focus on the grid spacing, topology and curvature because they also affect the aesthetics of the gridshell and are often established in the schematic phase of design.

2. METHODOLOGY

We measure the structural efficiency of pentagonal and hexagonal grids by calculating their failure load using the finite element method to analyse both linear and non-linear buckling. We study how their failure load changes with varying grid spacing for spherical cap shells of different span-to-height ratios. Lastly, we look at how sensitive the pentagon and hexagon topologies are to imperfections and compare them to the triangular and the quadrilateral grid.

2.1. Global and Local Geometry

We conduct our study on the global geometry of a spherical cap because a spherical cap is a common geometry used in gridshells [9]. Also, it is the same geometry used in previous work that studies triangular and quadrilateral topologies; the same study to which we will compare our pentagonal and hexagonal results [7].

The spherical cap is defined by its span L (30.5 m) and height h (1.5 m, 2.8 m, 5 m) from which we can calculate the radius R and angle of openness α (Fig. 1). In this study, we consider three different span-to-height ratios L/h= 20,11,6. The grid topologies are pentagons and hexagons and their member lengths are approximately equal when projected onto a plane (Fig. 2). The grid spacing s ranges approximately from 0.75 m to 3 m. Due to the circular boundary, the topology becomes irregular at the boundary. Lastly, the grid members are solid, square

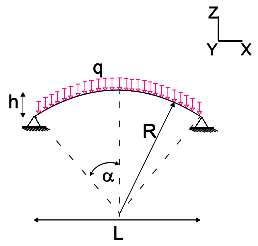

Fig. 1. Geometry, boundary and loading conditions of the spherical cap.

cross-sections of 127 mm x 127 mm. Note that we are only interested in the ratios of quantities and for the square section the two principal second moments of area are equal to each other and the ratio of the St. Venant torsional constant to the polar second moment of area is 0.1406 x 6 = 0.84 [10].

2.2. Boundary and Loading Conditions

The spherical cap is simply supported around the base and is uniformly loaded with vertical point loads at the joints. Asymmetric loading will be considered in future work as the hexagonal grid is expected to be more sensitive to nonuniform loading. However, due to the large number of parameters already in this study, we chose to limit the load type variation. The joints are rigidly connected, thus allowing for the transfer of bending and torsional moments.

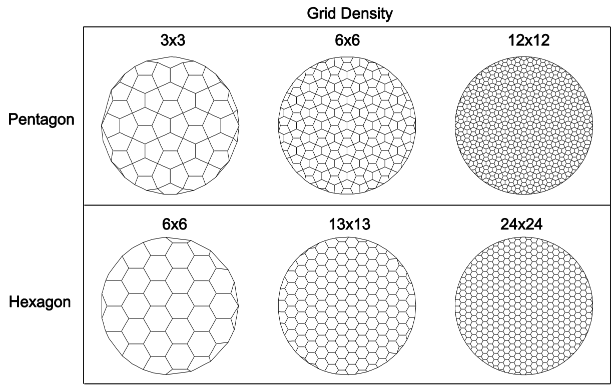

Fig. 2. Illustration of pentagonal and hexagonal topologies and density.

2.3. Material Properties

We consider a steel gridshell with an elastic-plastic profile where Young's Modulus E=200 GPa, strain hardening modulus E_t=345 MPa, yield stress σ_y=345 MPa, Poisson's ratio ν=0.3, and density ρ=7850 kg/m^3.

2.4. Finite Element Model

The finite element models consist of beam elements and are analyzed in the commercial finite element package ADINA. An example of the finite element model for the hexagonal grid with L/h=6 and spacing s=1.5 m is shown in Fig. 3.

A linear buckling analysis is done first to assess the overall behavior of the structure, but also to provide the shape of the initial imperfection for the nonlinear analysis [11]. The nonlinear analysis assumes large displacements and rotations, but small strains and uses a Load-Displacement-Constraint method where a load multiplier is used to increase and decrease the loading. The initial imperfection is the first buckling mode calculated in the linear buckling analysis scaled such that the maximum deviation from the sphere is 3 mm. All finite element modeling techniques are validated with benchmark problems and strain energy convergence tests [12].

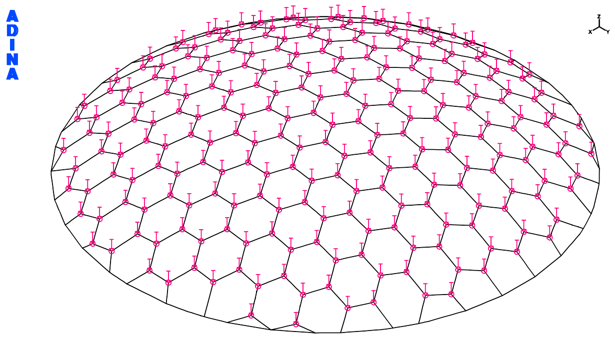

Fig. 3. Finite element model of the hexagonal spherical cap.

3. RESULTS

The linear and nonlinear analyses of the pentagon and hexagon grids are presented first. Next, the pentagon and hexagon grids are compared to a previous research done with triangle and quadrilateral grids [7].

3.1. Linear Buckling Analysis

The linear or eigenvalue buckling loads for the pentagon and hexagon topologies are plotted in Fig. 4. The ordinate is the log to the base 10 of the linear buckling load as a vertical load per unit area and the abscissa is the grid spacing. The lines represent the three different span-to-height L/h ratios.

As can be seen in the figure, the failure loads drop significantly between the triangular and the quadrilateral topology. The pentagonal grid is similar in structural capacity to that of the quadrilateral grid. The hexagonal grid is significantly less than the pentagonal. For example, for L/h=6 and a spacing s=0.75 m, the pentagon's capacity is almost twice that of the hexagon.

3.2. Structural Efficiency

There is a trade-off in the structural capacity of the shell and the volume of the members of the shell: the pentagon grid has more members than the hexagon. We measure this trade-off by normalizing the failure load to the weight of the structure, both measured as a force per unit area. This structural efficiency metric η is given by eq. 2 indicates how many times its own weight the structure can carry.

$$\eta = \frac{q_{cr}}{self\ weight} \qquad (2)$$

Note that we are here concerned with the relative efficiency of the different grid topologies. Other factors come into the efficiency; in particular one would expect solid sections to be less efficient than hollow members.

Fig. 5 plots the structural efficiency metric η as the ordinate and the grid spacing as the abscissa. It can be seen that for the most shallow shell L/h=6 there is little difference in structural efficiency between the pentagon and the hexagon grids. However, as the shell become steeper, and the grid denser, the pentagon is structurally more efficient.

Fig. 5 also plots the structural efficiency for the triangular and quadrilateral topologies from previous work. All four topologies exhibit the same trend: as the grid becomes denser the structural efficiency increases significantly and on average doubles its efficiency. Note that the member cross-sectional size remains constant for all the densities, but if smaller members were used for the higher density, this would reduce the structural efficiency.

For the more shallow shell (L/h=20) there is an insignificant advantage in terms of efficiency between using any of the topologies; any of the four topologies can offer the same efficiency.

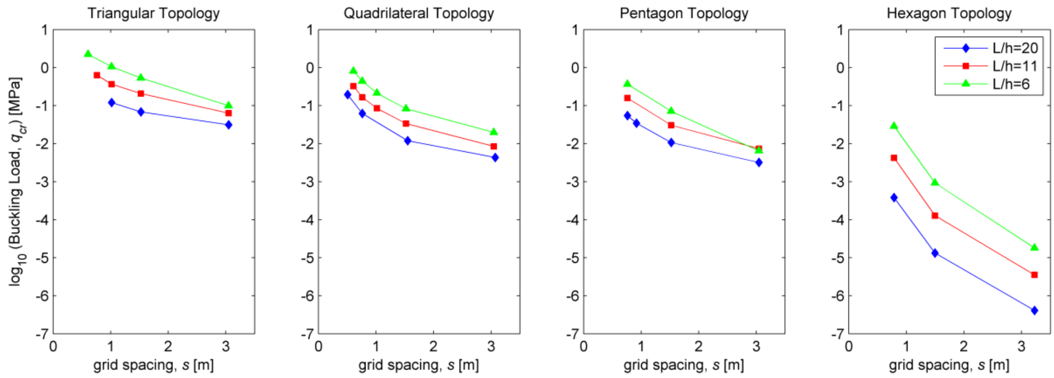

Fig. 4. The log base 10 linear buckling load for the four topologies (triangular, quadrilateral, pentagonal, hexagonal).

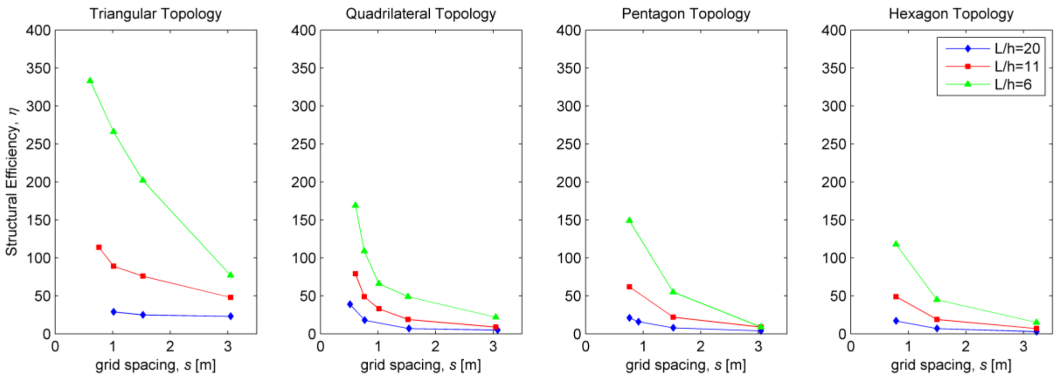

Fig. 5. The structural efficiency of the four topologies (triangular, quadrilateral, pentagonal, hexagonal).

3.3. Nonlinear Analysis

The nonlinear analysis accounts for prebuckling deflections and rotations and nonlinear material behavior. It also tells us whether the faster linear buckling analysis is valid and if the shell is sensitive to initial imperfections.

A nonlinear analysis was performed for each of the three span-to-height ratios, and for a grid spacing of s=1.5 m for both the pentagon and hexagon. The nonlinear analysis for the triangle and quadrilateral were also done as they were not done in the previous work [7]. An example of the load-displacement curve from the nonlinear analysis of the pentagon grid of L/h=6, s=1.5 m is shown in Fig. 6. As shown in the plot, the relatively large displacement prior to failure would imply that the pentagonal grid has a small sensitivity to imperfections. The decrease in the collapse load is due to imperfections and deflections prior to collapse. This was confirmed by repeating the nonlinear analysis with an elastic material. The same load-displacement curves were obtained as that of the nonlinear material.

In all cases the linear buckling analysis predicts a higher collapse load then the nonlinear analysis. Fig. 7 plots the percent decrease in the load prediction between the linear analysis and the nonlinear analysis.

The triangular grid and the hexagonal grid are the most sensitive to imperfections. They both exhibit a maximum decrease of 50% from the linear buckling load to the nonlinear. For the triangular grid the sensitivity to imperfections increases as the shell becomes shallower. However, the opposite is observed for the hexagon grid; as the shell become steeper the hexagon becomes more sensitive to imperfections.

The quadrilateral grid is the least sensitive to imperfections, and exhibits less than a 10% decrease in buckling load between the linear and the nonlinear analysis.It should be noted that while there is a significant decrease in the buckling load for the triangular grid, the triangle is still stronger and structurally more efficient than the quadrilateral, pentagon and hexagon.

4. DISCUSSION

There are non-structural advantages and disadvantages to using a pentagonal or hexagonal grid. Often the cost and difficulty in the design of the gridshell is the joint design. For a pentagonal grid, there is an irregularity in the number of members sharing a joint; there are either three or four members. However, for a hexagon there are uniformly three members sharing a joint.

The panel shape also provides a challenge to the cladding design. A triangular grid can be faceted with flat panels regardless of the curvature because three points define the plane. Panels with more than three edges can only achieve a flat or possibly single curved cladding if the geometry is constrained in some way [13, 14, 15].

The nonlinear analysis provides insight into the behaviour of the gridshell and whether it is truly behaving as a shell. The greater the sensitivity to imperfection the more the gridshell is behaving like a conventional shell. In the case of all the grids except the triangular, in-plane Vierendeel action is needed to provide some aspects of in-plane membrane stiffness.

The spherical cap analysed in this paper is properly supported as a shell and the shell is uniformly loaded. The results would change if the loading were non-uniform or if the shell were less well supported

5. CONCLUSION

This paper presents the first known analysis of Cairo-tiling (pentagonal) and hexagonal grids in the design of gridshells. Overall, pentagons perform within the same range as quadrilateral grids in terms of capacity and efficiency. Hexagonal grids do not provide as much structural capacity as the triangles, quadrilaterals, or pentagons, yet can be comparable in structurally efficiency. Future work will study the effect of asymmetric loading on pentagonal and hexagonal grids as they are expected to require more torsional stiffness.

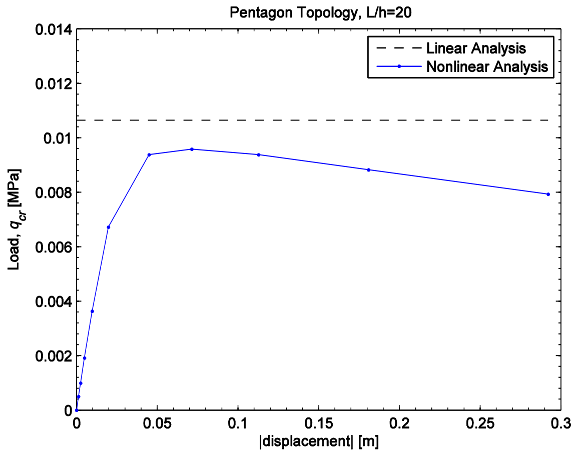

Fig. 6. The linear and nonlinear analysis for the pentagonal grid.

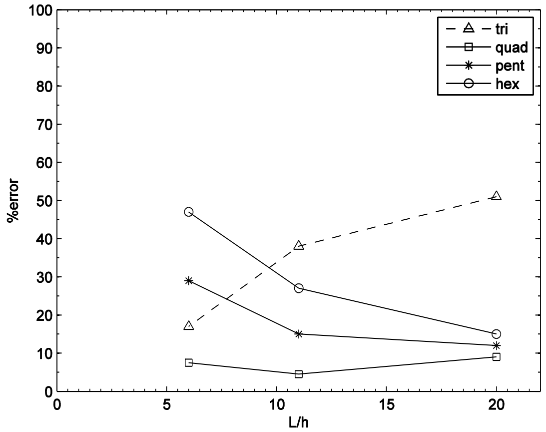

Fig. 7. The percent error between the nonlinear and linear analysis.

6. ACKNOWLEDGEMENTS

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