L’Approccio Frattale alla Meccanica non Locale
(The fractal approach to non-local mechanics)

Mario Di Paola
OUTLINES

• LATTICE THEORY

• GRADIENT AND STRONG NON-LOCAL THEORY

• PHYSICALLY-BASED APPROACH TO NON-LOCAL MECHANICS

• SELECTION OF THE DECAYING FUNCTION

• FRACTALS vs FRACTIONAL

• CONCLUSIONS
The Classical Continuum Mechanics (1D) Case

**Equilibrium of solid element:**

\[ N_{j+1} - N_j = f(x) \Delta V \]

\[ \Delta V = A \Delta x \]

\[ \frac{\Delta N_j}{A} = -f(x) \Delta x \quad \Delta x \to 0 \quad \frac{d\sigma(x)}{dx} = -f(x) \]

**Constitutive Equation (LOCAL)**

\[ \sigma = E\epsilon = E \frac{du}{dx} \]

**Governing equation of the 1D solid**

\[ \frac{d^2 u}{dx^2} = -\frac{f(x)}{E} \]

**Boundary conditions:**

\[ EA\epsilon_0 = -F_0 \quad ; \quad u(0) = u_0 \]

\[ EA\epsilon_L = F_L \quad ; \quad u(L) = u_L \]
The presence of microstructure in real-materials

RENORMALIZATION (WILSON 1972)

Continuum Mechanics Approach

Homogeneous and often isotropic elastic material
The Molecular Dynamics Approach

Each particle has three degree of freedom and its motion is ruled by the Newtons’ law: $\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t)$

- $\mathbf{u}(t) \in \mathbb{R}^{3n}$ Displacement vector
- $\mathbf{M} \in \mathbb{R}^{3n \times 3n}$ Mass matrix
- $\mathbf{K} \in \mathbb{R}^{3n \times 3n}$ Elastic bonds matrix

NANOSCALE $O(1-10 \ nm)$

MESOSCALE $O(0.001-1 \ \mu m)$

TERANUMBERS $O(10^{12})$

DEGREE OF FREEDOM INVOLVED !!!
The need for an enriched continuum mechanics

- In the late fifties and mid-sixties the basis of a generalized continuum mechanics had been proposed considering the inner microstructure.

The theory of micromorphic continuum
The Lattice Model of Materials (NN)

- Material properties often described at molecular level (Born-Von Karman):

  \[ F_j = m \ddot{u}_j + k \dot{u}_j + k (u_{j+1} - u_j) - k (u_j - u_{j-1}) \]

- A point-spring equivalent model:

  \[ k^l (u_{j+1} - 2u_j + u_{j-1}) = -F_j \]

  - Mass of Lattice Atoms
  - Lattice Elastic Constant
  - Lattice Distance

  \[ k^l = k = \frac{EA}{a} \]

  \[ F_j = \text{External load} \]
The Continuum Equivalence of the Lattice Models

- Derivation of the Waves Equation for an 1-D model

\[ \Delta x \to 0 \quad (n \to \infty) \]

\[ \frac{d^2 u}{dx^2} = -\frac{f(x)}{E} \]

- As \( \Delta x \to 0 \) it is implicitly assumed that the same kind of interaction exists at each smaller scale (EUCLIDEAN OBJECT).

- Lattice elements may exchange interactions still at distance \( a \) (NN).

\[ k \left( u_j - u_{j-1} \right) \quad \text{and} \quad k \left( u_{j+1} - u_j \right) \]

\[ \frac{EA}{\Delta x} \left( u_{j+1} - 2u_j + u_{j-1} \right) = -f_j A \Delta x \]

\[ k = \frac{EA}{\Delta x}; \quad m = \rho A \Delta x \]

\[ f(x) = \text{body force field} \]
The Non-Local Elasticity Theories

**GRADIENT (weak non-locality)**

\[ \sigma(x) = E_l \epsilon(x) + E_1 \frac{d}{dx} \epsilon(x) + E_2 \frac{d^2}{dx^2} \epsilon(x) + \ldots \]

- \( \sigma(x) \): Axial stress
- \( E_l, E_1, E_2 \): Elastic moduli
- \( \epsilon \): Axial strain

**INTEGRAL (strong non-locality)**

\[ \sigma(x) = E \epsilon(x) + \int g(x, \xi) \epsilon(\xi) d\xi \]

- \( g(x, \xi) \): Attenuation function
Essential references

Gradient non-local models


Integral non-local models


Fractal Mechanics


A Different approach

THE TEAM

Prof. Mario Di Paola, DISAG, Università di Palermo
Prof. Antonina Pirrotta, DISAG, Università di Palermo
Dr. Massimiliano Zingales, DISAG, Università di Palermo
Dr. Giuseppe Failla, MecMat, Reggio Calabria
Dr. Alba Sofi, Dip. Arte, Scienza e Tecnica del costruire, Reggio Calabria
Dr. Giulio Cottone, DISAG, Università di Palermo
Dr. Francesco Marino, MecMat, Università di Reggio Calabria
Dott. Gianvito Inzerillo, DISAG, Università di Palermo
The proposed model (Bounded domain)

\[ f(x_j) \]

\[ \Delta x \]

\[ V_h \]

\[ Q^{(h,j)} \]

\[ q^{(h,j)} = \text{sign}(x_h - x_j)(u_h - u_j) g(x_h, x_j) \]

\[ Q_j = \sum_{h=j+1}^{\infty} Q^{(h,j)} - \sum_{h=-\infty}^{j-1} Q^{(h,j)} \]


The proposed model (continue…)

\[ \sigma_I(x) = \frac{N(x)}{A} \]

\[ \Delta N_j + Q_j + f(x_j) A \Delta x = \Delta N_j + \sum_{h=j+1}^{m} q^{(h,j)} (A \Delta x)^2 - \sum_{h=1}^{j-1} q^{(h,j)} (A \Delta x)^2 + f(x_j) A \Delta x = 0 \]

\[ E \frac{d^2 u(x)}{dx^2} - A \int_0^L \left[ u(x) - u(\xi) \right] g(|x - \xi|) d\xi = -f(x) \]

\[ E \frac{d^2 u(x)}{dx^2} - A \int_{-\infty}^{\infty} \left[ u(x) - u(\xi) \right] g(|x - \xi|) d\xi = -f(x) \]
Mechanical interpretation of non-locality

**LOCAL MODEL**

\[
F(0) \quad f(x) \quad F(L) \quad F(0)
\]

\[
\sigma(x) = \frac{N(x)}{A}; \quad \varepsilon(x) = \frac{N(x)}{EA}
\]

\[
u^T = [u_1, u_2, \ldots, u_m]
\]

\[
f^T = [f_1, \ldots, f_m] \Delta x
\]

\[
K^l u = f
\]

\[
k^l = \frac{EA}{\Delta x} \quad \Delta x = \frac{L}{m}
\]

**Equilibrium of j**\(^{th}\) **node**

\[
\lim_{\Delta x \to 0} \left[ A \frac{\Delta^2 u(x_j)}{\Delta x} = -\frac{f_j A \Delta x}{E} \right] \Rightarrow \frac{d^2 u}{dx^2} = -\frac{f(x)}{E}
\]
Mechanical interpretation of non-locality II

**NON-LOCAL MODEL**

\[ K_{jh}^{nl} = A^2 \Delta x^2 g \left( |x_j - x_h| \right) \]

\[ j \neq h \]

\[ K_{jj}^{nl} = \sum_{h=1}^{m} K_{jh}^{nl} \quad \text{for} \quad h \neq j \]

\[ K = K^l + K^{nl} \quad \Rightarrow \quad K^{nl} = \begin{bmatrix} K_{11}^{nl} & -K_{12}^{nl} & -K_{13}^{nl} & \cdots & \cdots & -K_{1m}^{nl} \\ K_{21}^{nl} & K_{22}^{nl} & -K_{23}^{nl} & \cdots & \cdots & -K_{2m}^{nl} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{m1}^{nl} & \cdots & \cdots & \cdots & \cdots & K_{mm}^{nl} \end{bmatrix} \]

\[ K u = f \]
The stress-strain Relations and the overall Cauchy stress

\[
\sigma(x) = \frac{1}{A} \left( \sum_{j=2}^{m} K_{1j}^{nl} (u_j - u_1) + K_l (u_2 - u_1) \right) \quad 0 < x < \Delta x
\]

\[
\sigma(x) = \frac{1}{A} \left( \sum_{j=3}^{m} K_{1j}^{nl} (u_j - u_1) + \sum_{j=2}^{m} K_{2j}^{nl} (u_j - u_2) + K_l (u_3 - u_2) \right) \quad \Delta x < x < 2\Delta x
\]

GENERALIZING \quad r\Delta x < x < (r+1)\Delta x

\[
\sigma(x) = \frac{1}{A} \left( \sum_{j=r+1}^{m} \sum_{h=1}^{r} K_{hj}^{nl} (u_j - u_h) + K_l (u_{r+1} - u_r) \right) \quad \Delta x \to 0
\]

\[
= \left( E \frac{(u_r - u_{r-1})}{\Delta x} - A \sum_{j=r+1}^{m} \sum_{h=1}^{r} (u_h - u_j) g \left( \left| x_j - x_h \right| \right) (\Delta x)^2 \right)
\]
The stress-strain relations and the overall Cauchy stress (II)

\[ \sigma(x) = \frac{1}{A} \left( \sum_{j=r+1}^{m} \sum_{h=1}^{r} K_{nj}^{nl}(u_j - u_h) + K_l(u_{r+1} - u_r) \right) 
\quad r \Delta x < x < (r + 1) \Delta x \]

AT THE LIMIT
\[ \Delta x \rightarrow 0 \]

\[ \sigma(x) = E \frac{du}{dx} - A \int_{\xi_2 : x}^{\xi_1 : 0} \int_{\xi_2 : x}^{\xi_1 : 0} (u(\xi_1) - u(\xi_2)) g(|\xi_1 - \xi_2|) d\xi_1 d\xi_2 \]

\[ \sigma(x) = \sigma_l(x) + \sigma_{nl}(x) = \frac{1}{A} (N(x) + Q(x)) \]

\[ \sigma_l(x) = \frac{N(x)}{A} = E \frac{du}{dx} \]

\[ \sigma_{nl}(x) = \frac{Q(x)}{A} = -A \int_{\xi_2 : x}^{\xi_1 : 0} \int_{\xi_2 : x}^{\xi_1 : 0} (u(\xi_1) - u(\xi_2)) g(|\xi_1 - \xi_2|) d\xi_1 d\xi_2 \]
Comparisons between the Eringen model and the proposed model of long-range interactions

\[
\sigma(x) = E \varepsilon(x) + C \int_{a}^{b} \varepsilon(\xi) g(|x - \xi|) \, d\xi
\]

**ERINGEN (1972)**

\[
\sigma(x) = E \varepsilon(x) - A \int_{\xi_1=x}^{b} \int_{\xi_2=a}^{x} \left[ u(\xi_2) - u(\xi_1) \right] g(|\xi_1 - \xi_2|) \, d\xi_1 d\xi_2
\]

**Mechanically-based (2008)**

**UNBOUNDED DOMAINS**

**(POWER-LAW, HELMOLTZ)**

\[
g_{K}(|x_j - x_m|) = C \exp\left(-\frac{|x_j - x_m|}{\lambda}\right)
\]

\[
\sigma(x) = E \varepsilon(x) + C \lambda^2 \int_{-\infty}^{\infty} A \varepsilon(\xi) \exp\left[-\frac{|x - \xi|}{\lambda}\right] \, d\xi
\]

Comparisons between the Gradient and the proposed model of long-range interactions

\[ E \frac{d^2 u(x)}{dx^2} - A \int_{-\infty}^{\infty} \left[ u(x) - u(\xi) \right] g(|x - \xi|) d\xi = -f(x) \]

Taylor series expansion of \( u(x) \) about location \( x \)

\[ E \frac{d^2 u(x)}{dx^2} - \sum_{j=1}^{\infty} r_{2j} \frac{d^{2j} u(x)}{dx^{2j}} = -f(x) \quad u(x) \in C_\infty \]

\[ r_{2j} = \frac{A}{2j!} \int_{-\infty}^{\infty} (\xi - x)^{2j} g(|x - \xi|) d\xi \]

The elastic problem of the 1D Continuum with long-range forces

\[
\sigma(x) = E \frac{du}{dx} - A \int_{\xi_2 : x}^{\xi_1 : 0} \left( u(\xi_1) - u(\xi_2) \right) g(|\xi_1 - \xi_2|) d\xi_1 d\xi_2
\]

Constitutive

\[
\frac{d\sigma(x)}{dx} = \frac{d}{dx} \left( \sigma_l(x) + \sigma_{nl}(x) \right) = -f(x)
\]

Equilibrium

\[
\frac{du}{dx} = \varepsilon(x)
\]

Compatibility

BOUNDARY CONDITIONS

\[
u(0) = u_0 \quad u(L) = u_L
\]

Kinematic

\[
A\sigma(x)|_L = A\left( \sigma_l(x)|_L + \sigma_{nl}(x)|_L \right) = N(x)|_L = F
\]

Static

\[
A\sigma(x)|_0 = A\left( \sigma_l(x)|_0 + \sigma_{nl}(x)|_0 \right) = N(x)|_0 = -F
\]

The Distance-Decaying function

\[ E \frac{d^2 u}{dx^2} - A \int_0^L (u(x) - u(\xi)) g(|x - \xi|) d\xi = f(x) \]

\[ EA(0) \varepsilon(0) = A\sigma_i(0) = -F_0; \quad EA(L) \varepsilon(L) = A\sigma_i(L) = F_L \]

Local Cauchy stress equilibrates the external tractions

- The decaying function must be symmetric and must belong to the class of monotonically decreasing function of the arguments as from lattice theory.

\[ g(x, \xi) = g(\xi, x) = g(|x - \xi|) \]

- One-Dimensional Geometry (Euclidean)
  - Helmoltz
  - Bi-Helmoltz
  - Kröner-Eringen model for unbounded solid
The decaying function: The Fractional Problem

• Fractional Power-Law: \[ g \left( |x - \xi| \right) = \frac{c_\alpha E}{\Gamma(1-\alpha)} \frac{1}{|x - \xi|^{1+\alpha}} \quad 0 \leq \alpha \leq 1 \]

• The Fractional Differential Problem:

\[
\frac{d^2 u(x)}{dx^2} - c_\alpha \left[ (\hat{D}^\alpha_{0+} u)(x) + (\hat{D}^\alpha_{L-} u)(x) \right] = -\frac{f(x)}{E}
\]

• Integral Parts of Marchaud Fractional Derivative

\[
(\hat{D}^\alpha_{a^+} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int^x_a f(x) - f(\xi) \frac{1}{(x - \xi)^{1+\alpha}} d\xi
\]

\[
(\hat{D}^\alpha_{b^-} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int^b_x f(x) - f(\xi) \frac{1}{(\xi - x)^{1+\alpha}} d\xi
\]

Unbounded Domains

\[
(\hat{D}^\alpha_+ f)(x) = (\hat{D}^\alpha_{+\infty} f)(x) \quad ; \quad (\hat{D}^\alpha_- f)(x) = (\hat{D}^\alpha_{-\infty} f)(x) \quad \quad a \to -\infty, b \to \infty
\]
Numerical application

Free-Free bar

- In real materials the strains **ARE NOT UNIFORM** in tensile specimen under uniform stress

\[
\frac{d^2 u(x)}{dx^2} - c_\alpha \left[ (\hat{D}_0^\alpha u)(x) + (\hat{D}_L^\alpha u)(x) \right] = -\frac{f(x)}{E}
\]

\[ E = 7200 \ \text{daN/mm}^2 \]

\[ F = 10 \ \text{KN} \]

\[ \alpha = 0.5 \]

\[ A = 100 \ \text{mm}^2 \]

\[ L = 200 \ \text{mm} \]

\[ c_\alpha = 1.40 \ \text{mm}^{\alpha-2} \]
The 3D Non-Local Elasticity: The long-range forces

The relative displacement

\[ \eta_k(x, \xi) = u_k(x) - u_k(\xi) \]

The director vector

\[ r_k(x, \xi) = \frac{x_k - \xi_k}{\sqrt{(x_k - \xi_k)(x_k - \xi_k)}} \]

The directional Jacobi tensor

\[ G_{jk} = r_k r_j g(x, \xi) \]

The specific long-range force applied in a point \( x \)

\[ q(x, \xi) = G(x, \xi) \eta(x, \xi) \]

The 3D Non-Local Elastic Problem

Field Equations & boundary Conditions

Euler-Lagrange Equations

$$\mu^* \nabla^2 u_k (x) + (\lambda^* + \mu^*) u_{i,ik} (x)$$

$$+ \int_V g_{kj} (x,\xi) \eta_j (x,\xi) dV (\xi) = -\bar{b}_k (x) \quad x \in V$$

$$\left\{ \begin{array}{l}
u_k (x) = \bar{u}_k (x) \quad \forall x \in S_c \\
\sigma_{kj}^{(l)} (x) n_j = \bar{p}_{nk} (x) \quad \forall x \in S_f \\
\end{array} \right.$$
The 3D Non-Local Elastic Problem: Some Preliminary results

Displacement $u_1(x_1, -a/2)$ and $u_1(x_1, 0)$

Galerkin method

Finite difference method

Local theory, $\beta_1=1.0$

Local theory, $\beta_1=0.9$

$x_1 [\text{m}]$
The 3D Non-Local Elastic Problem: Some Preliminary results

Strain $\varepsilon_{1}(x_{1}, -a/2)$ and $\varepsilon_{1}(x_{1}, 0)$

Galerkin method

Finite difference method

Local theory, $\beta_{1}=1.0$

Local theory, $\beta_{1}=0.9$
Research Developments


The Fractal Mechanical Model: The NN Lattice

- It corresponds to a mechanical, point-spring model as:

\[
Q_P^{(j,j+1)} = K_P^{(j,j+1)} \left( u_{j+1}^{(p)} - u_j^{(p)} \right)
\]

\[
K_P^{(j,j+1)} = \frac{b_P A^2 l^2}{\Gamma(1-\beta)|x_{j+1} - x_j|^{\gamma}} = \frac{b_P \Delta V_P^2}{(l)^{\gamma}}
\]

\[
K_1^{(j,j+1)} = \frac{2^\gamma b_P \Delta V_1^2}{(l)^{\gamma}} = 2^{\gamma-2} K_P^{(j,j+1)}
\]

\[
K_2^{(j,j+1)} = \frac{3^\gamma b_P \Delta V_2^2}{(l)^{\gamma}} = 3^{\gamma-2} K_P^{(j,j+1)}
\]

\[
K_h^{(j,j+1)} = h^{\gamma-2} K_P^{(j,j+1)}
\]
The Fractal Mechanical Model: The HB Dimension

\[ K_{P}^{(j,j+1)} = \frac{b_{p} \Delta V_{1}^{2}}{(l)^{\gamma}} \]

\[ \Delta V_{h} = l / hA \]

\[ K_{h}^{(j,j+1)} = \frac{b_{p} \left( \Delta V_{h} \right)^{2}}{(l/h)^{\gamma}} = h^{\gamma-2} K_{P}^{(j,j+1)} \]

\[ \phi_{h} = \frac{1}{2} K_{h}^{(j,j+1)} \left( \frac{l}{h} \right)^{2} = \frac{b_{p} A^{2}}{2} \left( \frac{l}{h} \right)^{4-\gamma} \]

- Elastic potential energy invariance at any observation scale
  \[ d_{H} = \frac{1}{4-\gamma} \]

\[ [\Phi_{h}]^{s} = h[\phi_{h}]^{s} = h(h^{\gamma-4})^{s} [\Phi_{0}]^{s} = [\Phi_{0}]^{s} \]

\[ 0 < \gamma < 4 \]

The Governing Operators

- Horizontal Equilibrium of the Generic Element of the NN lattice:

\[
\begin{align*}
K_p^{(j,j-1)} (u_j^{(p)} - u_{j-1}^{(p)}) & \quad \text{for } F_j = 0 \\
K_p^{(j,j+1)} (u_{j+1}^{(p)} - u_j^{(p)}) & \quad \text{for } F_j = 0 \\
K_h^{(j,j+1)} (u_j^{(h)} - 2u_{j}^{(h)} + 2u_{j-1}^{(h)}) & = -f_j A \frac{l}{h}
\end{align*}
\]

Local Euclidean mechanics \( \gamma = 3 \)

\[
b_p A \frac{d^2 u}{dx^2} = -f(x)
\]

Local Fractal mechanics \( \gamma \neq 3 \)

\[
b_p A \frac{\Delta^2 u}{\Delta x^{\gamma-1}} = -f(x)
\]

- The classical governing equation of the continuum mechanics have been obtained without introducing contact, local, stress in the model. We argue that contact stress is obtained as the resultant of short-range forces between adjacent particles of solids.
The MultiScale Mechanical Fractal (MSF)

• The interaction distance do not change as we refine the observation scale.

\[ K_{P}^{(j,j+1)} = \frac{b_{P} \Delta V_{P}^{2}}{(l)^{\gamma}} \]

\[ K_{1}^{(j,j+1)} = 2^{-2} K_{P}^{(j,j+1)} \]

\[ K_{2}^{(j,j+2)} = 3^{-2} K_{P}^{(j,j+1)} \]

\[ K_{2}^{(j,j+3)} = 3^{-2} K_{P}^{(j,j+1)} \]
The MultiScale Mechanical Fractal: The Scaling Law

- Physical interactions became negligible but cannot vanish beyond distance $l$ so that they are mathematically zero only as $l \to \infty$

The MSF is obtained as the union of self-similar elastic chains as $n \to \infty$

$$M_F = \lim_{n \to \infty} \bigcup_{j=1}^{n} p_j E^{(j)}$$

It maintains its self-similar nature at any resolution scale and the scaling law of the springs between different levels:

$$K_{h}^{(j, j+i)} = \frac{h^{\gamma-2}}{i^{\gamma}} \frac{b_{p} A_{p}^{2} l^{2}}{l^{\gamma}} = \frac{h^{\gamma-2}}{i^{\gamma}} K_{p}^{(j, j+1)}$$

$$d_H = \frac{1}{4-\gamma} \quad \implies \quad 0 < \gamma < 4$$
The MSF Model: ENERGY INVARIA NCE

- The Elastic potential energy at the $h=1$ scale

$$\Phi_p = \frac{1}{2} K_p^{(j,j+1)} l^2 \quad \Phi_1^s = \sum_{i=1}^{n} \left[ \phi_0^{(j,j+i)} \right]^s = \sum_{i=1}^{n} \left[ \frac{1}{i^{\gamma-2}} \Phi_p \right]^s$$

- The Elastic potential energy at the $r=1/n$ scale

$$\Phi_h^s = \sum_{i=1}^{n} n \left[ \phi_h^{(j,j+i)} \right] = \sum_{i=1}^{n} n \left[ \frac{n^{\gamma-4}}{i^{\gamma-2}} \Phi_p \right]^s$$

THE INVARIANCE CONDITION

$$\Phi_1^s = \sum_{i=1}^{n} \left[ \phi_0^{(j,j+i)} \right]^s = \sum_{i=1}^{n} \left[ \frac{1}{i^{\gamma-2}} \Phi_p \right]^s = \sum_{i=1}^{n} \left[ \frac{n^{\gamma-4}}{i^{\gamma-2}} \Phi_p \right]^s = \sum_{i=1}^{n} \left[ \phi_h^{(j,j+i)} \right] = \Phi_h^s$$

The HB dimension of the mechanical MSF

$$d_H = \frac{1}{4-\gamma}$$
Q: What about operators?

- Equilibrium equations at the \( n \) observation level:

\[
F_j = f_j A \frac{l}{n}
\]

\[
- \sum_{i=-\infty}^{j-1} K_n^{(j,j+i)} (u_j^{(n)} - u_{j+i}^{(n)}) + \sum_{p=r+1}^{\infty} K_n^{(j,j+i)} (u_{j+i}^{(n)} - u_j^{(n)}) = -F_j
\]

\[
K_n^{(j,j+i)} = \frac{b_p A^2}{(il/n)^\gamma} \left( \frac{l}{n} \right)^2
\]

\[
- \sum_{i=-\infty}^{j-1} b_p A^2 \left( \frac{u_j^{(n)} - u_{j+i}^{(n)}}{(x_j - x_{j+i})^\gamma} \left( \frac{l}{n} \right)^2 \right) + \sum_{i=j+1}^{\infty} b_p A^2 \frac{(u_{j+i}^{(n)} - u_j^{(n)})}{(x_{j+i} - x_j)^\gamma} \left( \frac{l}{n} \right)^2 = -f_j A \left( \frac{l}{n} \right)
\]

\[
\Delta x = l/n \to 0
\]

\[
b_p A \left[ \int_{-\infty}^{x} \frac{u(x) - u(\xi)}{(x-\xi)^\gamma} d\xi + \int_{x}^{\infty} \frac{u(x) - u(\xi)}{(x-\xi)^\gamma} d\xi \right] = f(x)
\]

A: Marchaud Derivatives!!

\[
b_p A \left[ \left( D^\gamma_+ u \right)(x) + \left( D^\gamma_- u \right)(x) \right] = -f(x)
\]

Q: What about operators?

A: Marchaud fractional derivative

\[
b_p A \left[ \int_{-\infty}^{x} \frac{u(x) - u(\xi)}{(x - \xi)^\gamma} d\xi + \int_{x}^{\infty} \frac{u(x) - u(\xi)}{(x - \xi)^\gamma} d\xi \right] = f(x)
\]

\[
b_p = \frac{\alpha c_\alpha}{\Gamma(1 - \alpha)}
\]

\[
\gamma = 1 + \alpha
\]

Marchaud Fractional derivatives if:

\[
0 < \alpha \leq 2
\]

Fractional Riesz-Weyl potential if:

\[
-1 < \alpha \leq 0
\]

Non-admissible for internal stress scaling:

\[
2 < \alpha < 3
\]

The role of the fractal dimension

\[ d_H = \frac{1}{4 - \gamma} \]

Simple mechanical fractal: Euclidean solids only with classical differential operators

Multiscale mechanical fractals: Fractional-order operators.

0 < \gamma < 4
The Euclidean case

\[ d_H = \frac{1}{4-\gamma} = \frac{1}{3-\alpha} \]

\[ \frac{\alpha c^a}{\Gamma(1-\alpha)} \left[ \int_{-\infty}^{x} \frac{u(x) - u(\xi)}{(x-\xi)^{1+\alpha}} d\xi + \int_{x}^{\infty} \frac{u(x) - u(\xi)}{(x-\xi)^{1+\alpha}} d\xi \right] = c_{\alpha} \left[ (D^a_+ u)(x) + (D^a_- u)(x) \right] = f(x) \]

\[ \gamma = 3 \quad \iff \quad \begin{cases} \quad d_H = 1 \\ \quad \alpha = 2 \end{cases} \quad \left[ c_{\alpha} = FL^{\alpha-4} \right] \]

\[ c_2 \left[ (D^2_+ u)(x) + (D^2_- u)(x) \right] = -2c_2 \frac{d^2u}{dx^2} = f(x) \]

The Equilibrium Equation of Cauchy solid

\[ E = 2c_2 \quad \Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{f(x)}{E} \]
Conclusions

• Solid bodies with fractal mass distributions may be studied within the Mechanically-Based model of Long-Range Interactions and some important conclusions may be withdrawn.

1. The introduction of fractal distribution of the mass density in the solid leads to a fractal mechanical model represented by a point-spring model whose stiffness is power-law decreasing with the interdistance.

2. The assumption that only interactions with adjacent particle is included in the model leads toward a simple mechanical fractal that seems to be ruled by the local version of fractional operators. The order of the operators is connected with the fractal dimension of the mechanical fractal model (study in progress).

3. Assuming that long-range interactions are maintained at any resolution scale and that interactions extends to infinity a Multiscale Fractal mechanical model is obtained. The fractal dimension of the MSF coincides with that of the composing elastic chains.

4. The governing operators of the Multiscale Fractal mechanical model are Marchaud-type fractional differential equations. Therefore we conclude that such operators rules the physics of multiscale fractal sets.
THANK YOU FOR THE ATTENTION!