STRESS CONCENTRATION

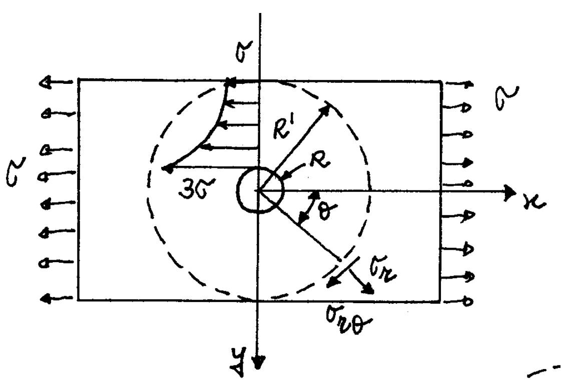
- Cracks always initiate at points of stress concentration.

- A crack, once initiated, becomes an <u>intense</u> stress concentrator itself.

- Two fundamental cases of plane elasticity:

INFINITE PLATE CONTAINING A CIRCULAR HOLE

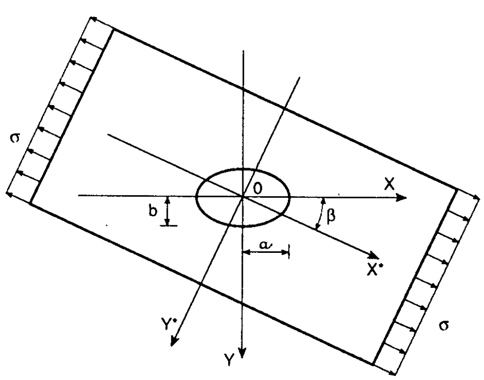
(Kirsh, G, (1898), V.D.I., 42, 797-807)



→ Stress Concentration

INFINITE PLATE CONTAINING AN ELLIPTICAL HOLE

(Kolosoff, G.V., *On an application of complex function theory to a plane problem of the mathematical theory of elasticity*, Yuriev, 1909;
Inglis, C.E., (1913), Stresses in a plate due to the presence of cracks and sharp corners,*Transactions of the Royal Institute of Naval Architectes*, 60, 219-241)



→ Stress Concentration

- In the limit of (minor axis) / (major axis) $\rightarrow 0$:

INFINITE PLATE CONTAINING A CRACK

(**Wieghardt**, K., (1907), *Z. Mathematik und Physik*, 55, 60-103; translated by Rossmanith (1995), On splitting and cracking of elastic bodies*, Fatigue Fract. Engrg. Mater. Struct.*, 18, 1371-1405)

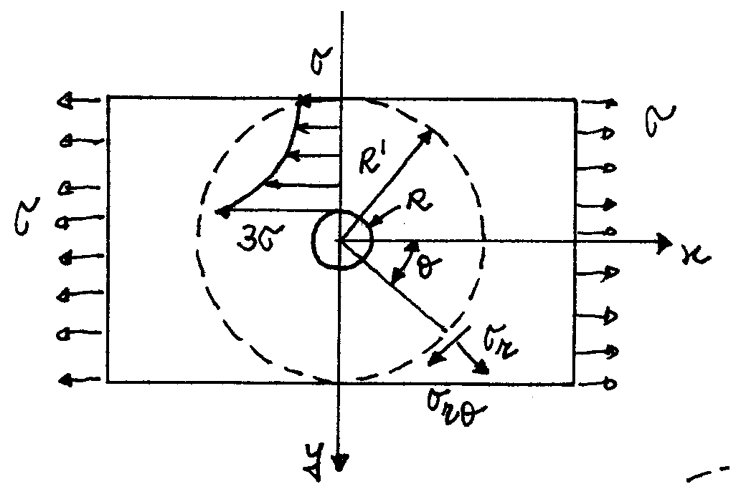
Muskhelishvili, N.I.*, Some basic problems of the mathematical theory of elasticity*, in Russian 1933 (in English 1953, Noorhoff-Groningen).
Westergaard, Bearing Pressures and cracks, (1937), *J. Applied Mechanics*, 6, A49-53.
Williams, M.L. (1952), Stress singularities resulting from various boundary conditions in angular corners of plates in extension, *J. Applied Mech*., 19, 526-528.
Williams, M.L. (1957), On the stress distribution at the base of a stationary crack, *J. Applied Mech.,* 24, 109-114.)

$\rightarrow$ Stress Intensification

INFINITE PLATE CONTAINING A CIRCULAR HOLE

(Kirsh, G, (1898), V.D.I., 42, 797-807)

- Consider infinite plate containing a circular hole of radius R and subject to a remote tensile stress $\sigma_x = \sigma$.



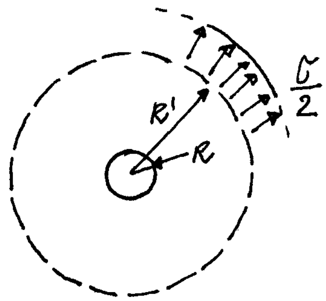
- Consider portion of plate within concentric circle of radius R' >> R so that stress field is not perturbed by hole (Saint- Venant's Principle)

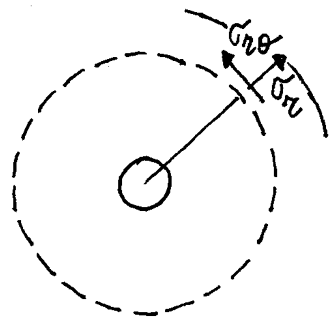
- Stress field at r = R' (Mohr's circle):

$$\sigma_r = \frac{\sigma}{2}(1+\cos 2\theta)$$

$$\sigma_{r\theta} = -\frac{\sigma}{2}\sin 2\theta$$

- Decompose problem into:





Problem (1)

$\sigma_r = \sigma / 2, \ \sigma_{r\theta} = 0$

Problem (2)

$\sigma_r = \frac{\sigma}{2}(\cos 2\theta), \sigma_{r\theta} = -\frac{\sigma}{2}\sin 2\theta$

- **Solution problem (1):**

thick cylindrical tube under tension (Lamè):

$$\sigma_r = -\frac{\sigma}{2}\frac{R^2 R'^2}{R'^2 - R^2}\frac{1}{r^2} + \frac{\sigma}{2}\frac{R'^2}{R'^2 - R^2}$$

$$\sigma_\theta = +\frac{\sigma}{2}\frac{R^2 R'^2}{R'^2 - R^2}\frac{1}{r^2} + \frac{\sigma}{2}\frac{R'^2}{R'^2 - R^2}$$

- For $R' \to \infty$

$$\sigma_r = +\frac{\sigma}{2}\left(1 - \frac{R^2}{r^2}\right)$$

$$\sigma_\theta = +\frac{\sigma}{2}\left(1 + \frac{R^2}{r^2}\right)$$

- **Solution problem (2):**

Assume Airy stress function:

$$\Phi = f(r)\cos 2\theta$$

Impose compatibility, $\nabla^4\Phi = 0$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \Phi = 0$$

Substitute stress function and get the ordinary differential equation to determine f(r):

$$\cos 2\theta \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{4}{r^2}\right)^2 f = 0$$

General solution:

$$f = Ar^2 + Br^4 + \frac{C}{r^2} + D$$

Boundary conditions:

$$\sigma_r(\mathrm{R'}) = +\frac{\sigma}{2}\cos 2\theta$$

$$\sigma_{r\theta}(\mathrm{R'}) = -\frac{\sigma}{2}\sin 2\theta$$

$$\sigma_r(\mathrm{R}) = 0$$

$$\sigma_{r\theta}(\mathrm{R}) = 0$$

Recall:

$$\sigma_r = +\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_\theta = +\frac{\partial^2 \Phi}{\partial r^2}$$

$$\sigma_{r\theta} = +\frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}$$

Substitute into general solution to get constants:

For $R' \to \infty$: $A = -\frac{\sigma}{4},\ B = 0, C = -\frac{\sigma}{4}R^4, D = \frac{\sigma}{2}R^2$

- Solution problems (1) + (2):

$$\sigma_r = \frac{\sigma}{2}\left(1 - \frac{R^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + 3\frac{R^4}{r^4} - 4\frac{R^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma}{2}\left(1 + \frac{R^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r_\theta} = -\frac{\sigma}{2}\left(1 - 3\frac{R^4}{r^4} + 2\frac{R^2}{r^2}\right)\sin 2\theta$$

For r = R:

$$\sigma_r = 0$$
$$\sigma_\theta = \sigma(1 - 2\cos 2\theta)$$
$$\sigma_{r\theta} = 0$$

Maximum stress: $\sigma_\theta = 3\sigma$ for $\theta = \pi/2, 3\pi/2$

Minimum stress: $\sigma_\theta = -\sigma$ for $\theta = 0, \pi$

Note:

- stress concentration factor = 3, independent of R
- compression for $-\pi/6 \le \theta \le \pi/6$ and $-5\pi/6 \le \theta \le 7\pi/6$
- circumferential stress for $\theta = \pi/2$:

$$\sigma_\theta = \frac{\sigma}{2}\left(2 + \frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)$$

⇒ hole has a localized character
for r = 4R: $\sigma_\theta = \sigma(1 + 0.04)$

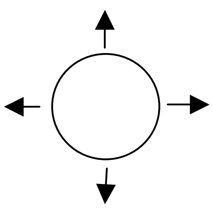
⇒ solution applicable to finite plates with width > 4R.

- stress field satisfies plane-strain and generalized plane stress.

- INFINITE PLATE WITH CIRCULAR HOLE SUBJECT TO BIAXIAL STRESS

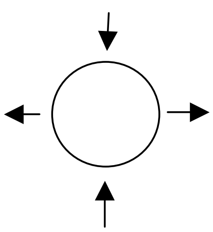
Apply superposition principle to get stresses at r = R:

1) Biaxial tension:



Uniform stress: $\sigma_\theta = 2\sigma$

2) Tension/compression (pure shear):



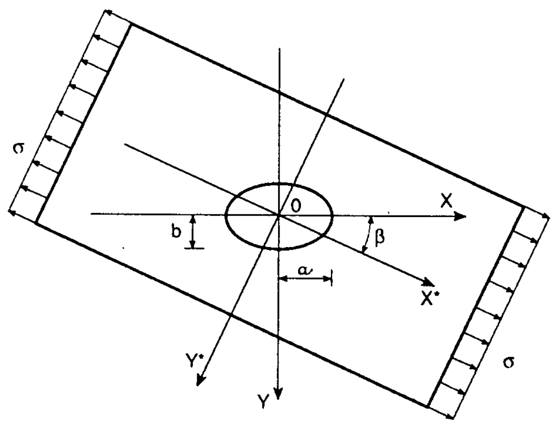
Maximum stress: $\sigma_\theta = 3\sigma + \sigma = 4\sigma$
for $\theta = \pi/2,\ 3\pi/2$

Minimum stress: $\sigma_\theta = -\sigma - 3\sigma = -4\sigma$
for $\theta = 0,\ \pi$

INFINITE PLATE CONTAINING AN ELLIPTICAL HOLE

(Kolosoff, G.V., On an application of complex function theory to a plane problem of the mathematical theory of elasticity, Yuriev, 1909;
Inglis, C.E., (1913), Stresses in a plate due to the presence of cracks and sharp corners,Transactions of the Royal Institute of Naval Architectes, 60, 219-241)

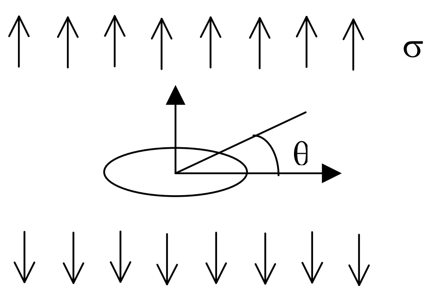
Hp: elliptical hole with: a = major axis, b = minor axis.



- Solution:

Kolosof (complex function theory);
Inglis (Conformal Mapping, elliptical coordinates)
(see Carpinteri, Meccanica dei materiali e della frattura, 1992, Pitagora, for details).

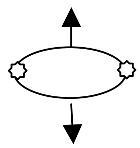
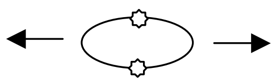
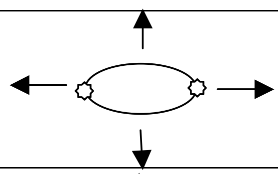
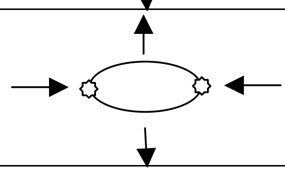
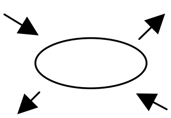
- For $\beta = \pi/2$ (tensile stress perpendicular to major axis):

 a = major axis, b = minor axis	$\sigma_\theta = \left(\frac{1 - m^2 - 2m + 2\cos 2\theta}{1 + m^2 - 2m\cos 2\theta} \right) \sigma$ m=(a-b)/(a+b).

Maximum circumferential stress:

$$\max \sigma_\theta = \sigma_\theta(0) = \left(1 + 2\frac{1+m}{1-m}\right)\sigma = \left(1 + 2\frac{a}{b}\right)\sigma$$

- Other cases:

	Tensile stress perpendicular to major axis	$\sigma_{max} = \sigma\left(1 + 2\frac{a}{b}\right)$
	Tensile stress perpendicular to minor axis	$\sigma_{max} = \sigma\left(1 + 2\frac{b}{a}\right)$
	Uniform stress	$\sigma_{max} = 2\frac{a}{b}\sigma$
	Pure shear	$\sigma_{max} = 2\sigma\left(1 + \frac{a}{b}\right)$
	Pure shear parallel to major and minor axes	$\sigma_{max} = \sigma\frac{(a+b)^2}{ab}$

- For a = b, solution for a circular hole (Kirsh);

- For *b/a* $\to$ 0 and tensile stress perpendicular to major axis:

<u>stress intensification;</u>

- For *b/a* $\to$ 0 and tensile stress perpendicular to minor axis: uniform stress equal to applied load,

no stress concentration

STRESS INTENSIFICATION

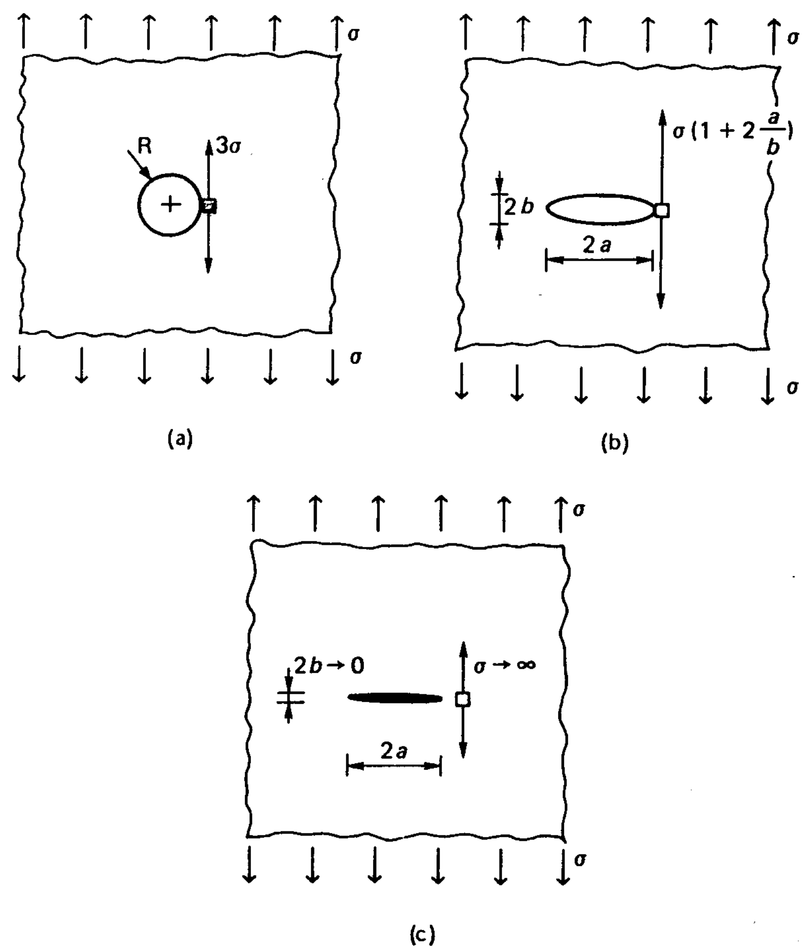


Plate with a crack: stress intensification at the crack tip

- What is the critical load in the cracked plate?

- What is the "fine structure" of the stress field at the crack tip?

- What is the power of the singularity?

(e.g., 2D problem of a point load acting on a semi-infinite plane (Kelvin): stresses = $O(r^{-1})$, displacements = $O(\log r)$)

Traditional design approach:

(2 parameters: $\sigma_{applied}$, σ_u)

$$\sigma_{applied} < \sigma_u$$

σ_u = yield or tensile strength

⇒ (a):	$\sigma_{applied} < \sigma_u / 3$	→ safe
⇒ (b):	$\sigma_{applied} < \sigma_u / (1+a/b)$	→ safe
⇒ (c):	$\sigma_{applied} \neq 0$	→ unsafe for any σ_y

???

Fracture Mechanics approaches in design:

(3 parameters: $\sigma_{applied}$, $\mathcal{G}_{IC}$ (or K_{IC}), a)

- **Energy Approach:**

the crack propagates when the energy available
for crack growth
overcomes the material resistance (fracture toughness)

- **Stress Intensity Factor Approach:**

the crack propagates when a local measure
of the singular stress field
reaches a critical value (fracture toughness)

- Energy Approach:

(**Griffith**, A.A., The phenomena of rupture and flow in solids*, Philosofical Transactions*, Series A, vol. 221, 1921, 163-198;

Griffith, A.A., The theory of rupture*, First Int. Congress of Applied Mechanics,* Delft, 1924, 55-63;)

- Stress Intensity Factor Approach:

(**Wieghardt**, K., (1907), *Z. Mathematik und Physik*, 55, 60-103; translated by Rossmanith (1995), On splitting and cracking of elastic bodies*, Fatigue Fract. Engrg. Mater. Struct.*, 18, 1371-1405)

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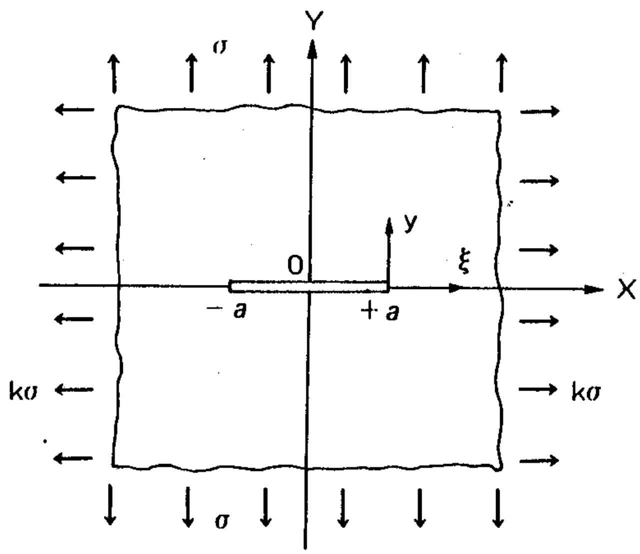
Williams, M.L. (1957), On the stress distribution at the base of a stationary crack, *J. Applied Mech.,* 24, 109-114.)

CRACK TIP SINGULARITY IN AN INFINITE PLATE CONTAINING A CRACK
<u>WESTERGAARD METHOD</u>

(**Westergaard**, Bearing Pressures and cracks, (1937), *J. Applied Mechanics*, 6, A49-53.)
(see Carpinteri, *Meccanica dei materiali e della frattura*, for details)



Hp:

homogeneous, isotropic, linearly elastic body
plane stress
crack length = 2a
biaxial load:

$$\sigma_y(z=\infty)=\sigma$$

$$\sigma_x(z=\infty)=k\sigma$$

with k = real constant, $-\infty \le k \le \infty$

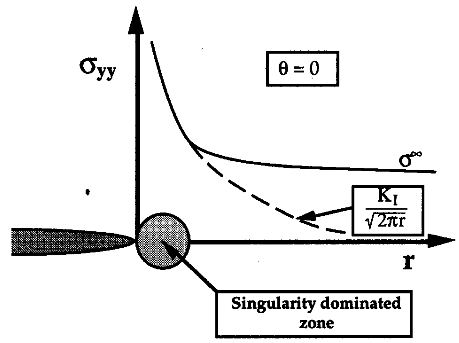
Problem: define stress field at crack tip

Stress field ahead of the crack tip in an infinite cracked plate (symmetry about x axis)

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + 2B$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$



Note:

- General expression for the stress components:

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta)$$

- inverse square root singularity in all stress components:

- power of the singularity and $f_{Iij}(\theta)$ depending on crack face boundary conditions and unaffected by remote boundary conditions

- stress field <u>univocally</u> defined by K_I (now still unknown):

→ K_I is a measure of the singular stress field
→ single parameter description of crack tip conditions

- K_I units: $[F]\,[L]^{-3/2}$

- Singularity dominated zone

Recall $g(a)/\sqrt{a} = K_I/\sqrt{\pi}$ then:

$$\boxed{K_I = \sigma\sqrt{\pi a}}$$

K_I = Stress Intensity Factor (SIF) in an infinite cracked plate subject to biaxial loading (symmetry about x axis):

Note:

- $\sigma_x(\infty)$ does not affect K_I.

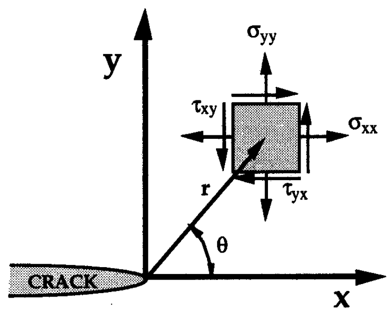
- K_I and stress field at the crack tip depend on crack length a

- $\sigma_x = \sigma(k-1)$ for $\theta = \pi$, along crack faces:

$\sigma_x = -\sigma$ uniaxial load (k = 0)

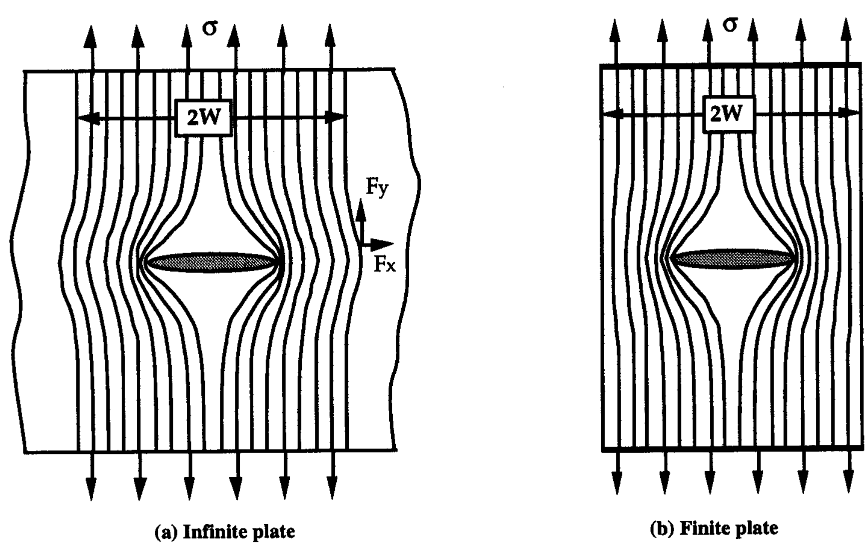
$\sigma_x = 0$ biaxial load, (k = 1 or $\sigma_x(\infty)=\sigma_y(\infty)=\sigma$)

EFFECT OF FINITE SIZE: MODE I PROBLEMS IN FINITE BODIES



- Stresses at the crack tip:

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta)$$



- Stress Intensity Factor in finite body:

$$K_I = \sigma\sqrt{\pi a}\,F$$

F dimensionless function, typically a polynomial expression, depending on <u>geometry</u> and <u>loading conditions</u>.

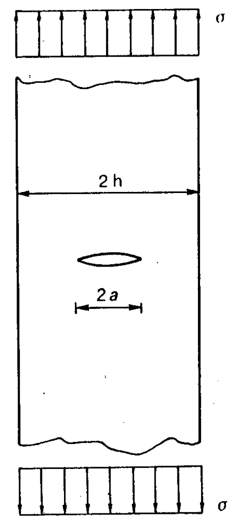
Examples:

- Cracked strip in tension:

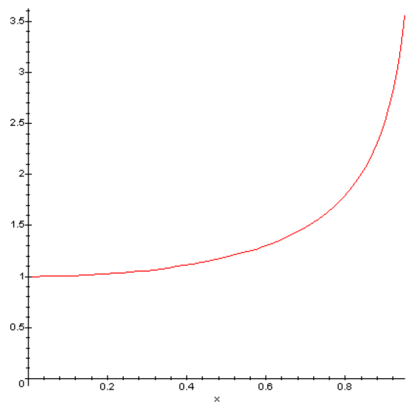
$$K_I = \sigma\sqrt{\pi a}\left(\sec\frac{\pi a}{2h}\right)^{1/2}$$

$$F = F\left(\frac{a}{h}\right) = \left(\sec\frac{\pi a}{2h}\right)^{1/2}$$

guess based on numerical solution



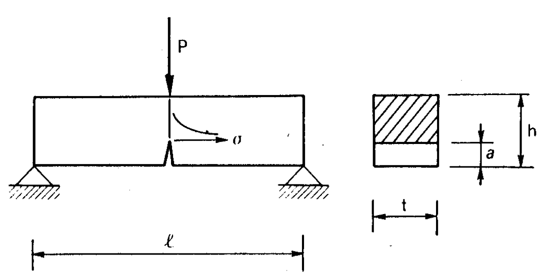
$F\left(\frac{a}{h}\right)$



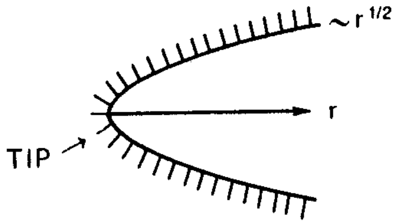
- Three point bending beam:

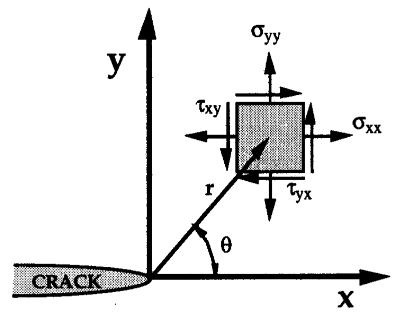
$$K_I = \frac{Pl}{th^{3/2}}F\left(\frac{a}{h}\right)$$

$$F\left(\frac{a}{h}\right) = 2.9\left(\frac{a}{h}\right)^{1/2} - 4.6\left(\frac{a}{h}\right)^{3/2} + 21.8\left(\frac{a}{h}\right)^{5/2} - 37.6\left(\frac{a}{h}\right)^{7/2} - 38.7\left(\frac{a}{h}\right)^{9/2}$$



CRACK OPENING DISPLACEMENT





- From plane stress elasticity:

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

then:

$$u_y = \int \varepsilon_y \, dy + \text{rigid body motion terms}$$

- substituting the stresses in the singularity dominated region:

$$u_y = \int \varepsilon_y \, dy = \frac{1}{E}\int (\mathrm{Re}Z_I + y\mathrm{Im}Z_I' - B)\, dy - \frac{\nu}{E}\int (\mathrm{Re}Z_I - y\mathrm{Im}Z_I' + B)\, dy$$

- It is easy to check that the following u_y satisfies previous equation:

$$u_y = \frac{2}{E}\mathrm{Im}\overline{Z}_I - \frac{1+\nu}{E} y\mathrm{Re}Z_I - \frac{1+\nu}{E} By$$

- using the complex potential:

$$\overline{Z}_I = \frac{K_I}{\sqrt{2\pi}} 2\xi^{1/2} + B\xi + C$$

- and polar coordinates:

$$\overline{Z}_I = \frac{2K_I}{\sqrt{2\pi}} r^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) + Br(\cos\theta + i\sin\theta) + C$$

-so that:

$$\boxed{COD = u_y(\theta = \pi) - u_y(\theta = -\pi) = 4\left(\frac{2}{\pi}\right)^{1/2} \frac{K_I}{E} r^{1/2}}$$

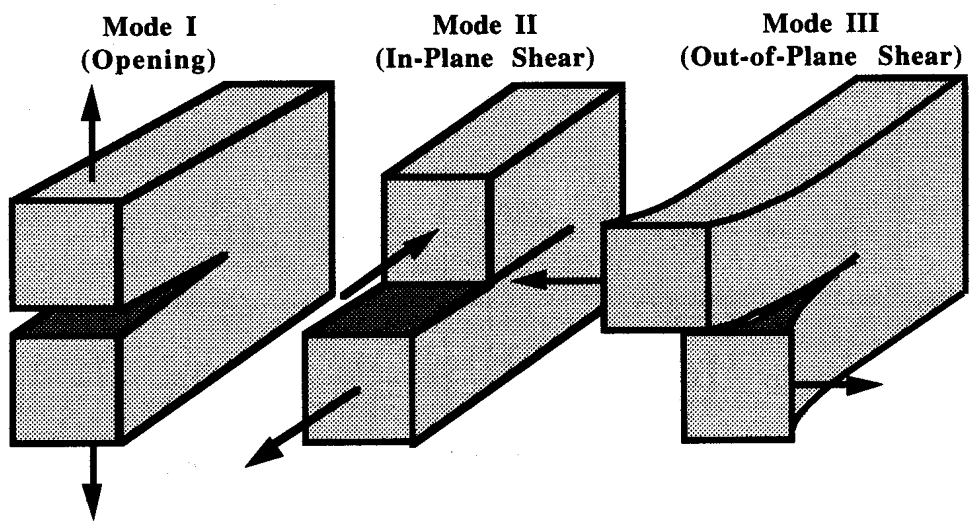
Note:

- COD depends on K_I/E and increases on increasing K_I/E

- COD is parabolic along the crack, COD $\approx r^{1/2}$

- COD has a vertical tangent in r = 0

ELEMENTARY MODES OF LOADING APPLIED TO A CRACK



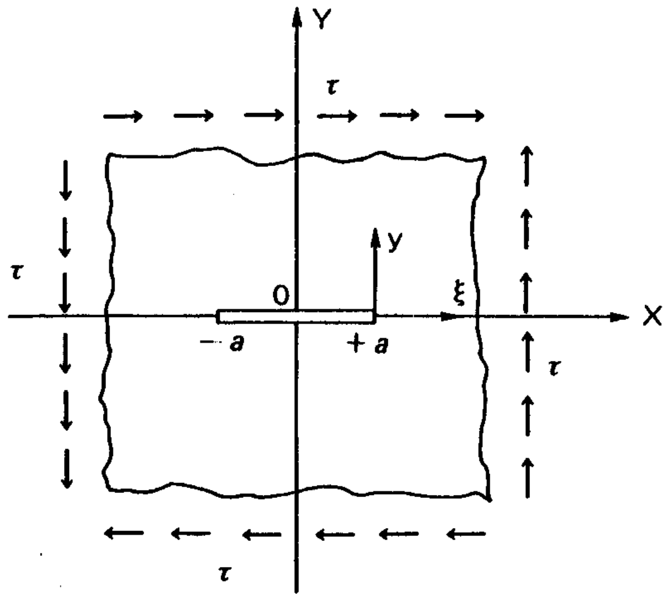
Mode I: Opening
symmetric loading about crack plane

Mode II: In-plane shear or in-plane sliding
anti-symmetric (or skew-symmetric) loading about x axis

Mode III: Tearing or out of plane shear
anti-symmetric (or antiplane) loading about x-z plane

INFINITE CRACKED PLATE IN MODE II

(**Westergaard**, Bearing Pressures and cracks, (1937), *J. Applied Mechanics*, 6, A49-53.)



- Airy stress function: $\Phi_{II} = -y\mathrm{Re}\overline{Z}_{II}$
- ………
- Stresses at crack tip:

$$\sigma_x = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

- $$\sigma_{ij} = \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$$

- Stress Intensity Factor:

$$K_{II} = \tau\sqrt{\pi a}$$

STRESS FIELD AHEAD OF THE CRACK TIP

Mode I:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + 2B$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

$\sigma_z = \nu(\sigma_x + \sigma_y)$ in plane strain, $\sigma_z = 0$ in plane stress

Mode II:

$$\sigma_x = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

$\sigma_z = \nu(\sigma_x + \sigma_y)$ in plane strain, $\sigma_z = 0$ in plane stress

Mode III:

$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

$$\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\sigma_x = \sigma_y = \sigma_{xy} = \sigma_x = 0$$

CRACK TIP DISPLACEMENT FIELDS

Mode I:

$$u_y = \frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\theta}{2}\left[(\kappa+1) - 2\cos^2\frac{\theta}{2}\right]$$

$$u_x = \frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}\,\cos\frac{\theta}{2}\left[(\kappa-1) + 2\sin^2\frac{\theta}{2}\right]$$

$$u_z = 0$$

Mode II:

$$u_y = -\frac{K_{II}}{2G}\sqrt{\frac{r}{2\pi}}\,\cos\frac{\theta}{2}\left[(\kappa-1) - 2\sin^2\frac{\theta}{2}\right]$$

$$u_x = +\frac{K_{II}}{2G}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\theta}{2}\left[(\kappa+1) + 2\cos^2\frac{\theta}{2}\right]$$

$$u_z = 0$$

Mode III:

$$u_z = -\frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}}\,\sin\frac{\theta}{2}$$

$$u_x = 0$$

$$u_y = 0$$

where:

$\kappa = 3-4\nu$ (plane strain)

$\kappa = (3-\nu)/(1+\nu)$ (plane stress)

SIF EVALUATION

- **Close form solutions:**

  complex function theory (conformal mapping, boundary collocation method, Laurent series expansion,…

  integral transforms (Fourier, Mellin, Hanckel transforms)

  eigenfunction expansion

limited to very simple cases

- **Computational solutions** (FEM, BEM, FDM, …)

- **Experimental solutions** (photoelasticity, moire interferometry,…)

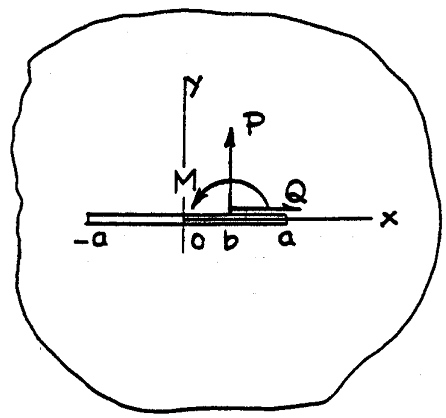
- Fracture handbooks

Tada, Paris and Irwin, (1985), The stress analysis of cracks handbook, Paris Prod., Inc., St. Luis (II edition).

Murakami (1987), Stress Intensity Factor handbook, Pergamon Press, NY.

Rooke and Cartwright, (1976), Compendium of Stress Intensity Factors, Her Majesty's Stationary Office, London.

Tada, Paris and Irwin, (1985), The stress analysis of cracks handbook



$$K_{I\,+a} = \frac{1}{2\sqrt{\pi a}}\left\{ P\sqrt{\frac{a+b}{a-b}} + \left(\frac{\kappa-1}{\kappa+1}\right)Q + \frac{Ma}{(a-b)\sqrt{a^2-b^2}} \right\}$$

$$K_{II\,+a} = \frac{1}{2\sqrt{\pi a}}\left\{ Q\sqrt{\frac{a+b}{a-b}} - \left(\frac{\kappa-1}{\kappa+1}\right)P \right\}$$

where

$$\kappa = \begin{cases} \dfrac{3-\nu}{1+\nu} & \text{plane stress} \\ 3-4\nu & \text{plane strain} \end{cases}$$

Method: Muskhelishvili's Method

Accuracy: Exact

References: Erdogan 1962, Sih 1962a

STRESS INTENSITY FACTOR
SUPERPOSITION PRINCIPLE

- For linear elastic materials individual components of stress, strain and displacement are additive.

- Similarly, stress intensity factors are additive:

$$K_I = K_I^{(A)} + K_I^{(B)} + K_I^{(C)} + \ldots$$

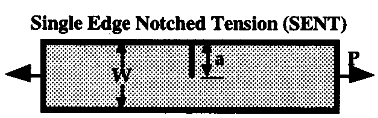
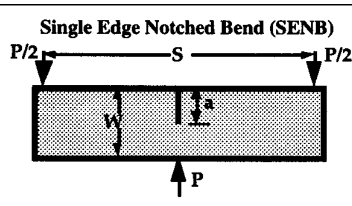
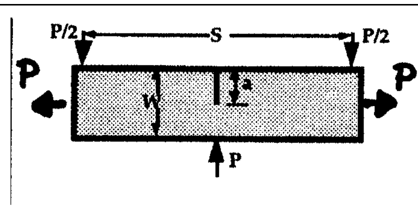
$$K_{II} = K_{II}^{(A)} + K_{II}^{(B)} + K_{II}^{(C)} + \ldots$$

$$K_{III} = K_{III}^{(A)} + K_{III}^{(B)} + K_{III}^{(C)} + \ldots$$

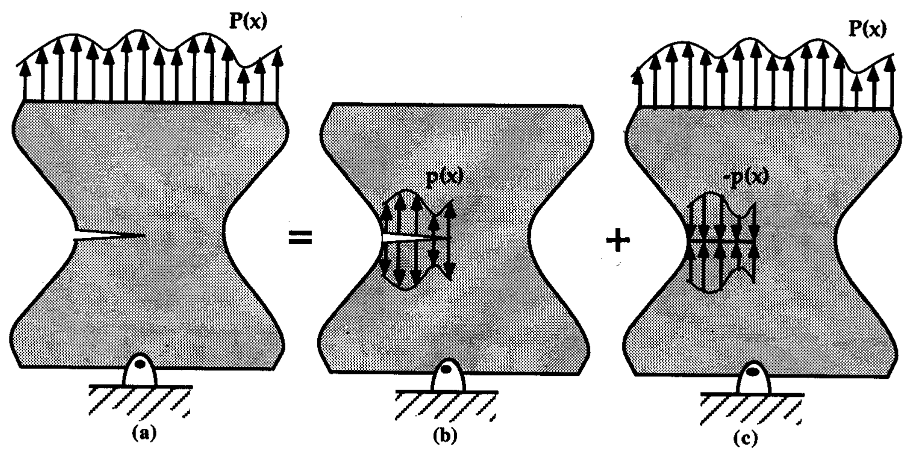
where:

$K_I^{(i)}, K_{II}^{(i)}, K_{III}^{(i)}$ (i = A, B, C, ….) are mode I, mode II and mode III stress intensity factors for the ith applied load.

Example:

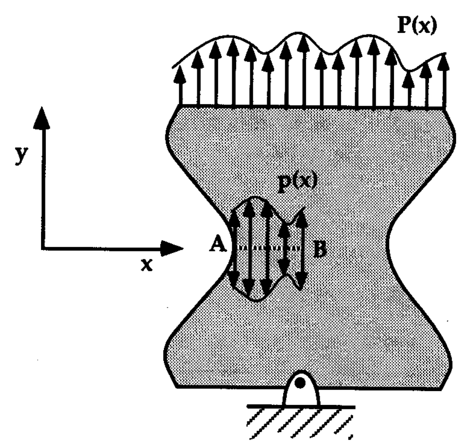
	$K_I = K_I^{(SENT)}$
	$K_I = K_I^{(SENB)}$
	$K_I = K_I^{(SENT)} + K_I^{(SENB)}$

SUPERPOSITION PRINCIPLE



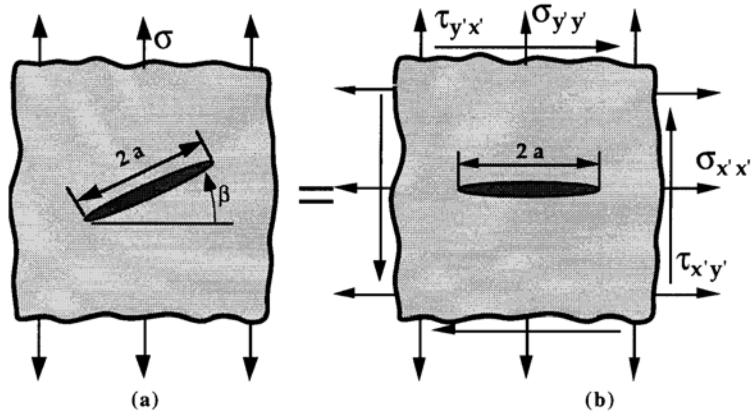
Any loading configuration can be represented by appropriate tractions applied directly to the crack faces

Proof:



SUPERPOSITION PRINCIPLE

- Infinite plate subject to uniaxial load with a crack of length 2a oriented at an angle β with the x axis



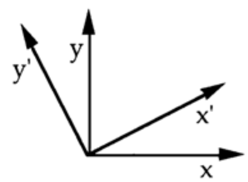


FIGURE 2.18 Through crack in an infinite plate for the general case where the principal stress is not perpendicular to the crack plane.

- For $\beta \neq 0$ the crack experiences Mode I and Mode II

- Introduce a new coordinate system $x' - y'$ with x' coincident with the crack orientation

- Define loads in the new system using Mohr's Circle

$$\sigma_{x'} = \sigma \sin^2\beta$$

$$\sigma_{y'} = \sigma \cos^2\beta$$

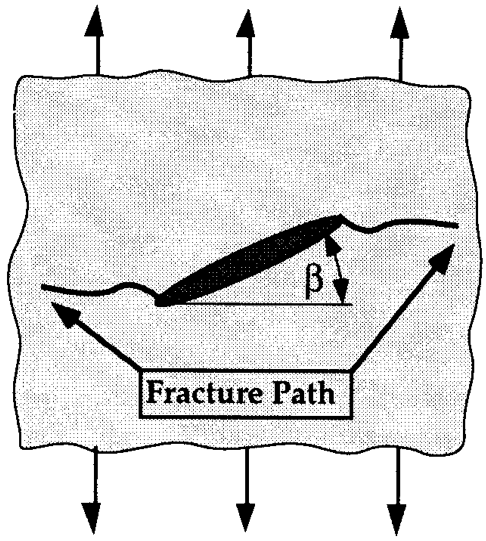
$$\sigma_{x'y'} = \sigma \sin\beta \cos\beta$$

- Stress intensity factors

$$K_I = \sigma \cos^2\beta\sqrt{\pi a}$$

$$K_{II} = \sigma \sin\beta\cos\beta\sqrt{\pi a}$$

MIXED MODE FRACTURE



Assumptions:

homogeneous, isotropic, linear elastic material
plane problem (Mode I + Mode II)

Griffith's Energy Criterion applies only to collinear crack growth

A Criterion for Mixed Mode fracture must define:

a) the direction of crack growth

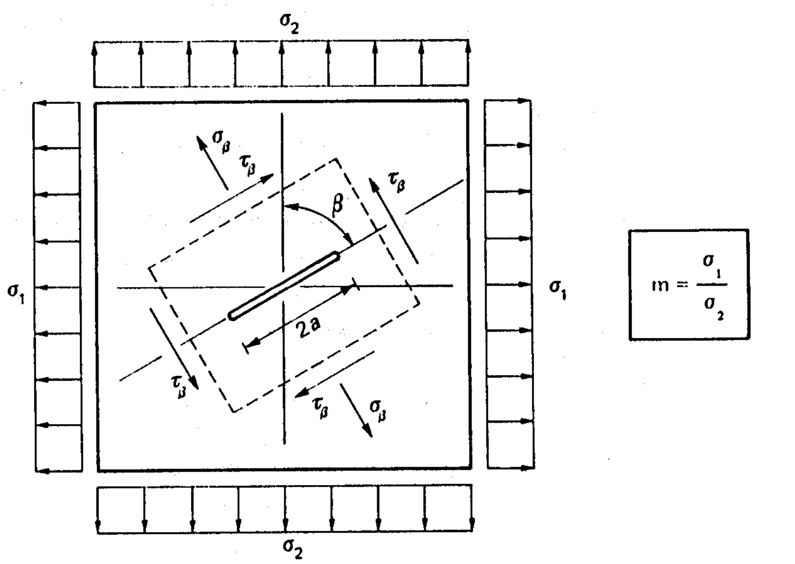
b) the critical values of the fracture parameters

MAXIMUM CIRCUMFERENTIAL STRESS CRITERION

(**Erdogan** and **Sih**, (1963), On crack extension in plates under plane loading and transverse shear, J. Basic Engineering, 85, 519-527)

Example of Application

- Cracked plate subject to biaxial load σ_1 - σ_2



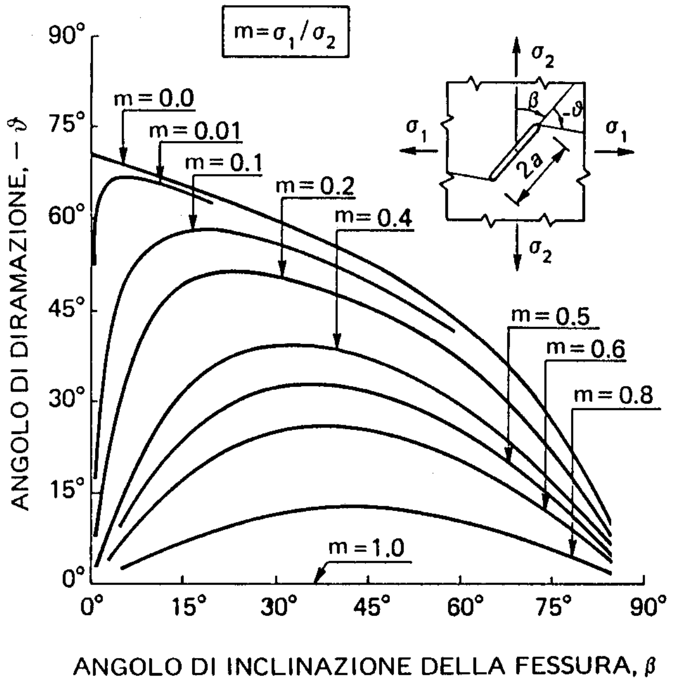
- Assume: $m = \sigma_1 / \sigma_2$
  β = crack direction about σ_2

- Crack tip stress intensity factors:

$$K_I = \sigma_\beta \sqrt{\pi a}$$
$$K_{II} = \tau_\beta \sqrt{\pi a}$$

Direction of crack growth

The crack propagates in a direction normal to the maximum circumferential stress σ_θ



Special cases:

1) $m = 1$, $\sigma_1 = \sigma_2$ $\Rightarrow$ Mohr's circle degenerates into a point
symmetry about crack direction

$$\sigma_\beta = \sigma_1$$
$$\tau_\beta = 0$$

$\Rightarrow K_{II} = 0, \quad \theta = 0$

$\Rightarrow$ collinear crack growth

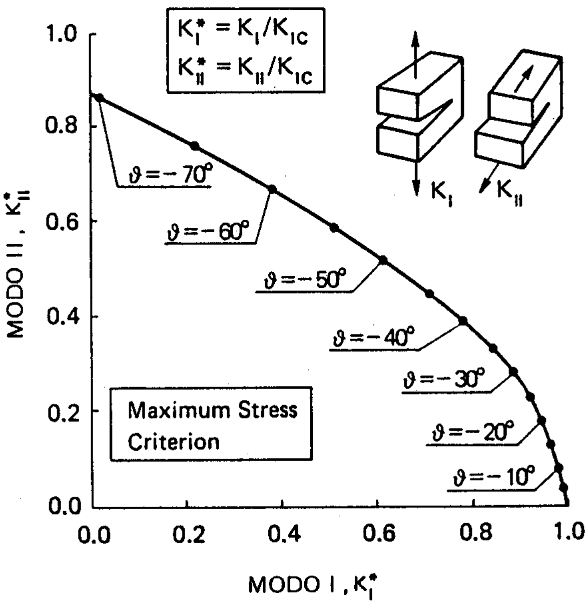
2) $m = 0$, $\sigma_1 = 0$ (uniaxial tension)
if $\beta = 0$ $\Rightarrow \theta = 0$ collinear crack growth

if $\beta \neq 0$ $\Rightarrow \lim_{\beta \to 0^+} \theta(m = 0, \beta) = 70.6°$

Criterion for crack growth

The Mixed-Mode crack will propagate along θ when $\sigma_\theta = \sigma_{\theta cr}$(Mode I)

Fracture locus



- the fracture locus is symmetric about the K^*_I axis

- the fracture locus is not defined for $K_I < 0$

GRIFFITH ENERGY CRITERION

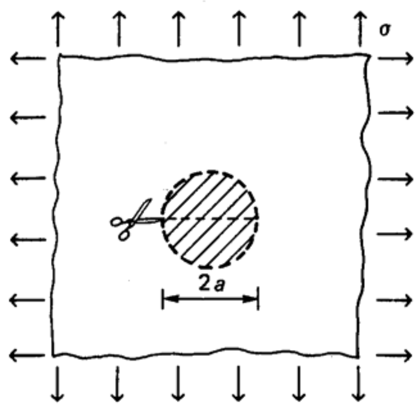
(**Griffith**, A.A., The phenomena of rupture and flow in solids*, Philosofical Transactions*, Series A, vol. 221, 1921, 163-198;
Griffith, A.A., The theory of rupture*, First Int. Congress of Applied Mechanics,* Delft, 1924, 55-63;)

- First law of Thermodynamics:
when a system goes from a nonequilibrium state to equilibrium there will be a net decrease in energy

→ A crack can form in a body only if such a process causes the total energy to decrease or remain constant (Griffith)

- Consider an infinite plate of unit width subject to biaxial load in plane stress conditions.

Imagine to create a crack of length 2a while keeping the remote displacements constant



The elastic strain energy released is proportional to the strain energy contained in a circle of radius a (after Inglis):

$$W_e = \pi a^2 \frac{\sigma^2}{E}$$

The strain energy, W, of the cracked plate is then given by:

$$W = W_0 - W_e = W_0 - \pi a^2 \frac{\sigma^2}{E}$$

where W_0 is the strain energy of the uncracked plate.

- The energy required to create the crack surface is:

$$W_s = 4a\gamma_s$$

- An incremental crack extension da is possible if:

$$\frac{dW}{da} + \frac{dW_s}{da} \leq 0$$

$$2\pi a \frac{\sigma^2}{E} \geq 4\gamma_s$$

$$\boxed{\sigma \geq \sqrt{\frac{2\gamma_s E}{\pi a}}}$$

(substitute E with E' for plane strain)

Fracture Energy:

$$\mathcal{G}_{IC} = 2\,\gamma_s$$

= energy required to create a unitary crack area.
(crack area = 2a; surface area = 4a)

- measure of the fracture toughness of the material

- to be defined through standard tests

$$\Downarrow$$

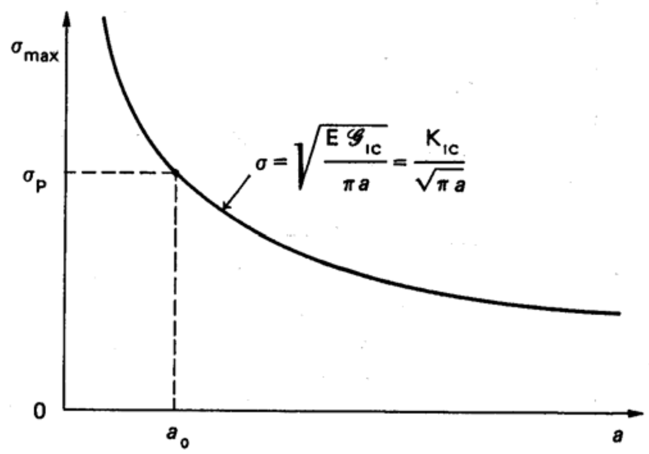
Griffith equation:

Stress for equilibrium crack growth:

$$\sigma_{max} = \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$$

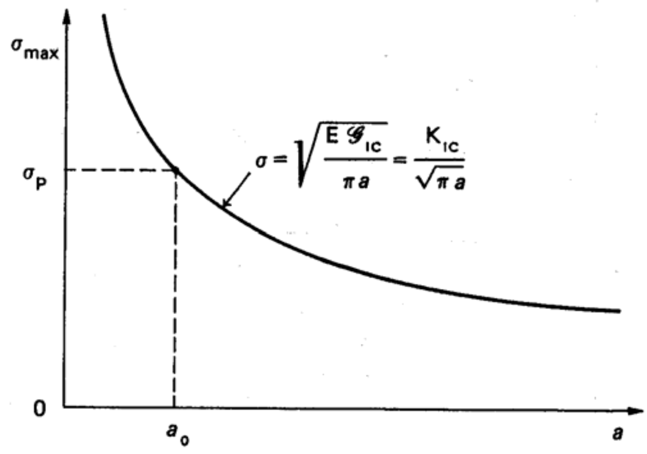
$\sigma > \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$ dynamic growth

$\sigma < \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$ no growth



Stress for equilibrium
crack growth:

$$\sigma_{max} = \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$$



$\sigma > \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$ dynamic growth

$\sigma < \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}}$ no growth

- crack growth is unstable: the load decreases on increasing the crack length

- two asymptotes:

• for $a \to \infty$, $\sigma_{max} \to 0$

• for $a \to 0$ $\sigma_{max} \to \infty$????

analogy with buckling collapse vs. strength collapse:

for $a<a_0$ the strength collapse precedes the fracture collapse

- The crack length a_0 corresponding to $\sigma_{max} = \sigma_y$ ($= \sigma_p$) is an equivalent measure of the microcracks, flaws and defects of the virgin material:

$$a_0 = \frac{1}{\pi} \frac{\mathcal{G}_{IC} E}{\sigma_y^2}$$

MODIFIED GRIFFITH EQUATION

(**Irwin**, (1948), Fracture Dynamics, Fracturing of Metals, ASM Cleveland, 146-166;
Orowan,(1948), Fracture and strength of solids, Reports on Progress in Physics, XII, p. 195.)

- In actual structural materials:

a) the energy needed to cause fracture is much higher than the surface energy (orders of magnitude higher)

b) inelastic deformations arise around the crack front $\rightarrow$ linear elastic medium with infinite stresses at the crack tip unrealistic

- Griffith equation can be modified to include the plastic work γ_P dissipated at the crack front

$$\boxed{\sigma \geq \sqrt{\frac{2E(\gamma_s + \gamma_p)}{\pi a}}}$$

- Orowan estimated $\gamma_P \approx 10^3 \gamma_s$ in typical metals.

- The modified Griffith equation is correct only if:

the size of the plastic zone around the crack tip is very small compared to the crack size (<u>small-scale yielding conditions</u>)

$\rightarrow$ the details of the crack tip stress do not affect the stress field in the elastic bulk of the medium
$\rightarrow$ <u>purely elastic solutions can be used to calculate the rate of energy available for fracture (!!!!!!!!)</u>

ENERGY BALANCE
IN BRITTLE FRACTURE

- Griffith criterion refers to a special case, i.e. infinite cracked plate, biaxial loading, fixed-grip conditions

- In general, the First Law of Thermodynamics yields:

$$\frac{dE_T}{dA} = \frac{dW}{dA} + \frac{dW_s}{dA} \leq 0$$

A = crack area

E_T = total energy

W_s = energy required to create new crack surfaces

W = total potential energy W = U - L

U = potential strain energy

L = potential of external forces

- From the definition of fracture energy:

$$\frac{dW_s}{dA} = 2\gamma_s = \mathcal{G}_{IC}$$

- <u>Energy criterion for brittle crack growth</u> (Mode I):

$$\boxed{-\frac{dW}{dA} \geq \mathcal{G}_{IC}}$$

THE STRAIN ENERGY RELEASE RATE

(**Irwin**, (1957), Analysis of stresses and strains near the end of a crack traversing a plate. ASME Journal of Applied Mechanics, 24, 361-364)

- Irwin introduced the Strain Energy Release Rate

$$\boxed{\mathcal{G} = -\frac{dW}{dA}}$$

as the energy available for an increment of crack extension, given by the total potential energy released due to the formation of a unit crack area

- $\mathcal{G}$ is also called Crack Driving Force or Crack Extension Force

- Energy criterion for brittle crack growth (Mode I):

$$\boxed{\mathcal{G}_I = \mathcal{G}_{IC}}$$

ENERGY APPROACH VS. STRESS APPROACH (CRITICAL CONDITIONS)

- The strain energy release rate is the fracture parameter describing the global behavior of the body.

- The stress intensity factor if the fracture parameter describing the local behavior (stresses, strains and displacements near the crack tip) of the body

- Stress criterion for brittle crack growth (Mode I):

$$\boxed{K_I \geq K_{IC}}$$

K_{IC} = critical stress intensity factor

K_{IC} is a measure of the material fracture toughness to be defined through standard tests

- $\mathcal{G}_{IC}$ and K_{IC} are related through a fundamental relationship

$$\boxed{\mathcal{G}_{IC} = \frac{K_{IC}^2}{E}}$$

derived from:

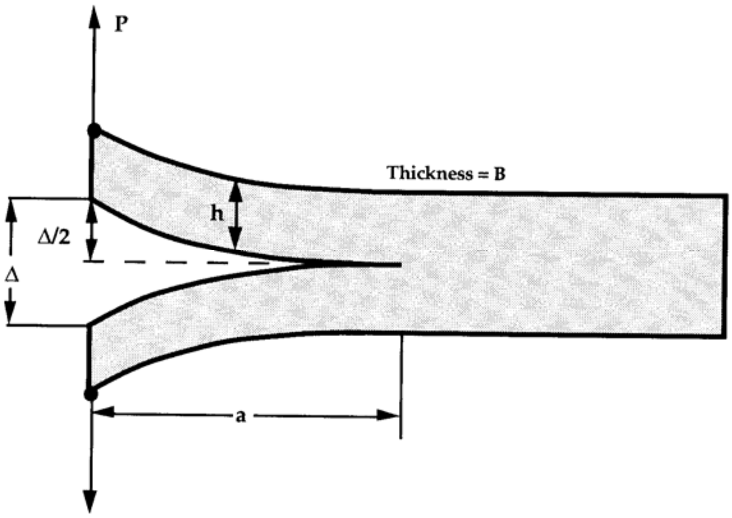
$$\sigma \geq \sqrt{\frac{\mathcal{G}_{IC} E}{\pi a}} \quad \text{energy instability condition}$$

$$\sigma \geq \frac{K_{IC}}{\sqrt{\pi a}} \quad \text{stress instability condition}$$

$$\Downarrow$$

$$\boxed{K_{IC} = \sqrt{\mathcal{G}_{IC} E}}$$

STRAIN ENERGY RELEASE RATE IN A DCB SPECIMEN



- Assume built-in end conditions for the two arms.

From beam theory:

$$\Delta/2 = \frac{Pa^3}{3EI} = \frac{4Pa^3}{EBh^3}$$

- The elastic compliance is:

$$C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

- The strain energy release rate in the DCB specimen is:

$$\mathcal{G} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B}\frac{6a^2}{3EI} = \frac{12\,P^2}{B^2}\frac{a^2}{Eh^3}$$

- Same conclusion from:

$$\mathcal{G} = -\frac{dW}{Bda} = -\frac{1}{B}\frac{d}{da}\left(-\frac{1}{2}P\Delta\right) = \frac{1}{B}\frac{d}{da}\left(\frac{P^2a^3}{3EI}\right) = \frac{P^2}{B}\frac{a^2}{EI} = \frac{12\,P^2}{B^2}\frac{a^2}{Eh^3}$$

- The critical load for crack propagation (Griffith criterion) is

$$P_{cr} = \frac{1}{a}\sqrt{\frac{\mathcal{G}_{IC} E h^3 B^2}{12}}$$

$\Rightarrow$ unstable crack growth

Note:

Strain energy release rate and critical load have been defined through a beam theory approximation.

If the geometry satisfies the assumptions of beam theory the solution is correct.

However, the local fields at the crack tip are not correctly given by beam theory

ENERGY RELEASE RATE IN A BODY SUBJECT TO DIFFERENT LOADING SYSTEMS

Hp: self-similar crack growth

- From the stress intensity factors of a body subject to different loading systems (A), (B), (C),…..

$$K_I = K_I^{(A)} + K_I^{(B)} + K_I^{(C)} + \dots$$
$$K_{II} = K_{II}^{(A)} + K_{II}^{(B)} + K_{II}^{(C)} + \dots$$
$$K_{III} = K_{III}^{(A)} + K_{III}^{(B)} + K_{III}^{(C)} + \dots$$

and the relationship between $\mathcal{G}$ and K

$$\mathcal{G} = \frac{K_I^2}{E} + \frac{K_{II}^2}{E} + \frac{K_{III}^2}{2G}$$

$\Rightarrow$ the strain energy release rate is

$$\mathcal{G} = \frac{(K_I^{(A)} + K_I^{(B)} + K_I^{(C)} + \dots)^2}{E} + \frac{(K_{II}^{(A)} + K_{II}^{(B)} + \dots)^2}{E} + \frac{(K_{III}^{(A)} + K_{III}^{(B)} + \dots)^2}{2G}$$

- The strain energy release rate contributions for each mode of fracture are addictive

$$\mathcal{G} = \mathcal{G}_{\mathrm{I}} + \mathcal{G}_{\mathrm{II}} + \mathcal{G}_{\mathrm{II}} = \frac{K_I^2}{E} + \frac{K_{II}^2}{E} + \frac{K_{III}^2}{2G}$$

- The strain energy release rate contributions for each loading system are not addictive, e.g. for mode I

$$\mathcal{G}_I^{(A)+(B)+(C)+\ldots} \neq \mathcal{G}_I^{(A)} + \mathcal{G}_I^{(B)} + \mathcal{G}_I^{(C)} + \ldots$$

NONLINEAR FRACTURE MECHANICS
THE COHESIVE CRACK MODEL

(**Dugdale**, D.S.: Yielding of steel sheets containing slits, *J. Mechanics Physics Solids* **8** (1960), 100-104
Barenblatt, G.I.: The formation of equilibrium cracks during brittle fracture. General ideas and hypotheses. Axially-symmetric cracks, *J. Applied Mathematics and Mechanics* **23** (1959), 622-636.
Barenblatt, G.I.: The mathematical theory of equilibrium cracks in brittle fracture, in H.L. Dryden and T. von Karman (eds.), *Advanced in Applied Mechanics*, Academic Press, New York, 1962, pp. 55-129.)

ductile materials ⇒ Dugdale's model

purely brittle materials ⇒ Barenblatt's model

quasi brittle materials (e.g. concrete) ⇒ Hillerborg's Fictitious Crack Model

brittle matrix composites ⇒ Bridged crack model

quasi brittle matrix composites ⇒ Cohesive crack model

Dugdale's Model for ductile fracture

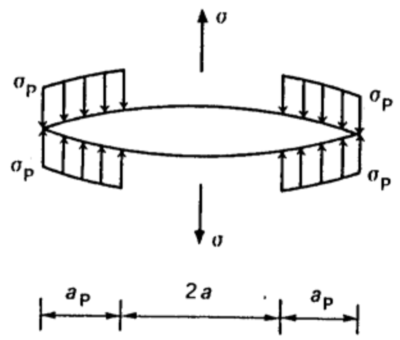
Fracture of elasto-plastic materials

Assumptions:

- Mode I crack in an infinite sheet under uniform tensile stress

- the material is ductile and the plastic deformations localized in a thin zone coplanar with the crack

- the plastic zone is modeled through a fictitious crack (of unknown length) and a uniform distribution of cohesive tractions σ_P = yield stress



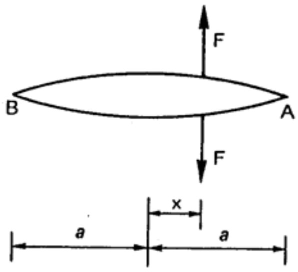
Length of the cohesive zone

The length of the cohesive zone a_P is calculated by imposing the condition of smooth closure of the crack faces:

$$K_I = K_{I\sigma} + K_{I\sigma_P} = 0$$

where: $K_{I\sigma} = \sigma\sqrt{\pi(a + a_P)}$

The SIF due to the plastic stresses is calculated using the SIFs due to a pair of concentrated forces F acting at x:



$$K_I(A) = \frac{F}{\sqrt{\pi a}}\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$K_I(B) = \frac{F}{\sqrt{\pi a}}\left(\frac{a-x}{a+x}\right)^{1/2}$$

so that:

$$K_{I\sigma_P} = -\frac{\sigma_P}{\sqrt{\pi(a+a_P)}}\int_a^{a_P}\left[\sqrt{\frac{(a+a_P)+x}{(a+a_P)-x}}+\sqrt{\frac{(a+a_P)-x}{(a+a_P)+x}}\right]dx$$

$$K_{I\sigma_P}(a) = -2\sigma_P\sqrt{\frac{(a+a_P)}{\pi}}\arccos\frac{a}{(a+a_P)}$$

From the condition for smooth closure:

$$\sigma\sqrt{\pi(a+a_P)} - 2\sigma_P\sqrt{\frac{(a+a_P)}{\pi}}\arccos\frac{a}{(a+a_P)} = 0$$

$$\frac{a}{a+a_P} = \cos\frac{\pi\sigma}{2\sigma_P}$$

In the limit $\sigma = 0 \quad \Rightarrow \quad a_P = 0$

$\sigma \to \sigma_P \quad \Rightarrow \quad a_P \to \infty$

Performing a Taylor series expansion (if $\sigma << \sigma_P$):

$$\frac{a}{a+a_P} = 1 - \frac{1}{2}\left(\frac{\pi\sigma}{2\sigma_P}\right)^2$$

$$\frac{a_P}{a+a_P} = \frac{\pi^2\sigma^2}{8\sigma_P{}^2}$$

and, as a function of $K_{I\sigma}$,

$$a_P = \frac{\pi}{8}\frac{K_{I\sigma}{}^2}{\sigma_P{}^2}$$

Length of the cohesive zone in critical conditions

Following Irwin's approach:

$$a_{PC} = \frac{\pi}{8}\frac{K_{IC}{}^2}{\sigma_P{}^2}$$

where: $$K_{IC} = \sqrt{\mathcal{G}_{IC} E}$$

and: $$\int_0^{\delta_a} \sigma_P \, d\delta = \sigma_P \delta_a = \mathcal{G}_{IC}$$

with δ_a the crack tip opening displacement given by:

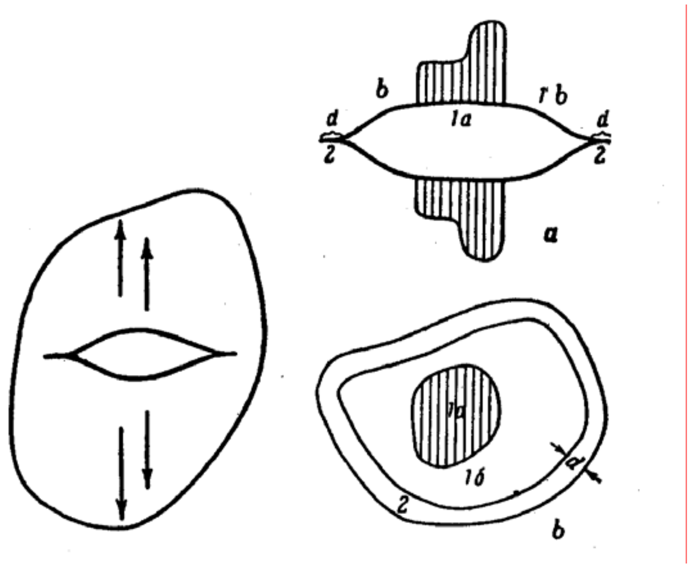
$$\delta_a = \frac{8\sigma_P a}{\pi E} \ln\left[\sec\left(\frac{\pi\sigma}{2\sigma_P}\right)\right] \quad \text{or} \quad \delta_a = \frac{K_I{}^2}{\sigma_P E} \quad (\text{if } \sigma << \sigma_P)$$

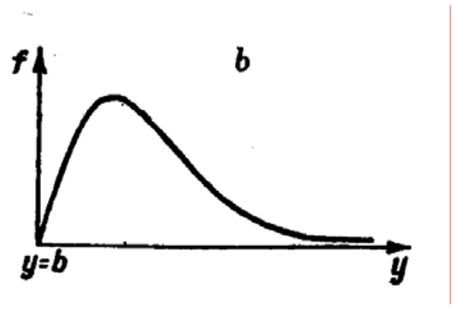
(use Castigliano's method)

Barenblatt's Model for purely brittle fracture

"In our work the question concerning equilibrium cracks forming during brittle fracture of a material is presented as a problem in the classical theory of elasticity, based on certain very general hypotheses concerning the structure of a crack and the forces of interaction between its opposite sides, and also on the hypothesis of **finite stresses at the ends of the crack**, or, which amounts to the same thing, the **smoothness of the joining of opposite sides of the crack at its ends**" (Barenblatt, 1959)

Consider a Mode I crack in a homogeneous, isotropic, linear-elastic infinite medium under uniform loading





Represent the atomic bonds holding together the two halves of the body separated by the crack as cohesive forces acting along the edge regions of the crack and attracting one side of the crack to the other

b = interatomic distance (order = 10^{-7} mm)

maximum intensity = ideal strength $\sigma_c \approx \dfrac{E}{\pi} \approx \sqrt{\dfrac{E\gamma_s}{b}}$

Determination of the cohesive forces

The accurate determination of the cohesive forces acting along the edge regions is difficult

- Assumptions:

1) The dimension of the edge region is small in comparison with the size of the whole crack

2) The displacements in the edge region, for a given material under given conditions, is always the same for any acting load

⇒ during crack propagation the edge region simply translate forward

1) + 2) = small scale yielding assumption

3) The opposite sides of the crack are smoothly joined at the ends or, which amounts to the same thing, the stress at the end of a crack is finite

⇒ K = 0 (zero stress intensity factor)

These assumptions lead to the definition of the critical state of mobile equilibrium which depends on the modulus of cohesion depending on the integral of the cohesive tractions along the edge region

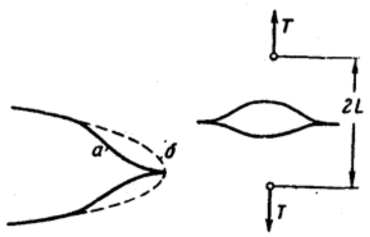
Equivalence of Griffith and Barenblatt approaches

Willis (by means of the complex variable method) and Rice (by means of the J integral) proved that:

Barenblatt's theory based on atomic forces is equivalent to Griffith's energy approach provided the integral of the cohesive forces is equal to the fracture energy

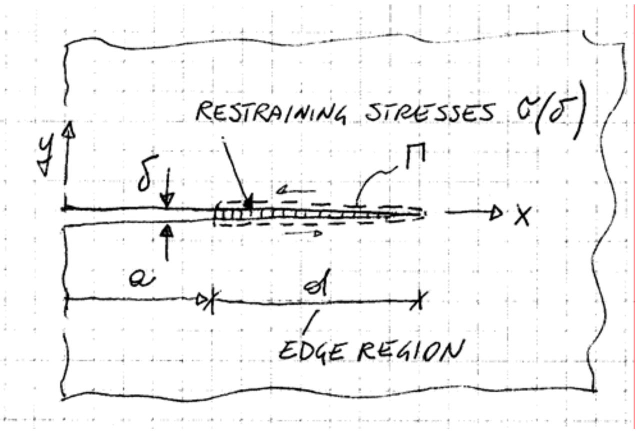
$$\int_0^{y^*} f\,dy = \mathcal{G}_{IC}$$

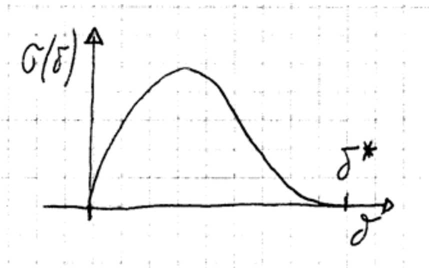
Barenblatt also showed that the cohesive forces essentially have effect only on the displacement field close to the edge of the crack and not on those in the main part of the crack



Rice's proof of the equivalence between Barenblatt's model and Griffith's energy approach

- Assume the crack to be in a state of mobile equilibrium
- Consider cohesive forces $\sigma(\delta)$ acting along the crack
- Let δ^* to be the separation distance beyond which the cohesive tractions vanish





- Consider the path Γ for which:
dy = 0, ds = dx on Γ^- and ds = -dx on Γ^+ , $T_1 = 0$, $T_2 = \sigma(\delta)$ on Γ^+ , $T_2 = -\sigma(\delta)$ su Γ^-

-Evaluate the J integral along Γ:

$$J = \int_{\Gamma} \left(U_d dy - T_i \cdot \frac{\partial u_i}{\partial x} ds \right) = -\int_{\Gamma^-} T_2 \frac{\partial u_2}{\partial x} ds - \int_{\Gamma^+} T_2 \frac{\partial u_2}{\partial x} ds = -\int_{a}^{a+d} -\sigma(\delta) \frac{\partial u_2}{\partial x} (dx) - \int_{d+a}^{a} \sigma(\delta) \frac{\partial u_2}{\partial x} (-dx) =$$

$$= -\int_{a}^{a+d} \sigma(\delta) \frac{\partial}{\partial x} (u_2^{+} - u_2^{-}) dx = -\int_{a}^{a+d} \sigma(\delta) \frac{\partial \delta}{\partial x} dx \quad \Rightarrow$$

$$J = \int_{0}^{\delta^*} \sigma(\delta) d\delta$$

$\Rightarrow$ The value of J which will cause crack extension is given by the integral of the cohesive forces:

$\Rightarrow$

$$\int_{0}^{\delta^*} \sigma(\delta)\, d\delta = \mathcal{G}_{IC}$$