BASI TEORICHE DEL METODO DEGLI ELEMENTI FINITI (MEF)

DOCENTE
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Elasticità

Elettromagnetismo

Fluidodinamica

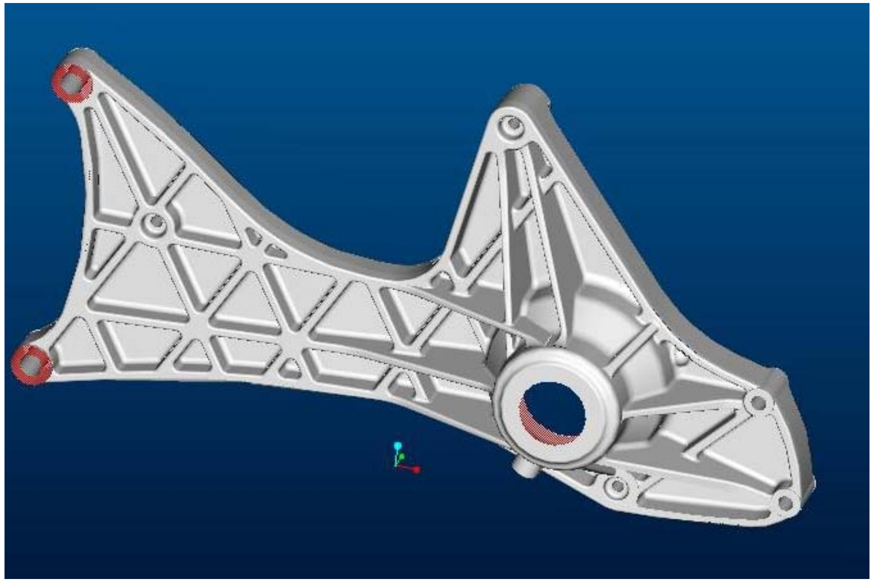
Termodinamica

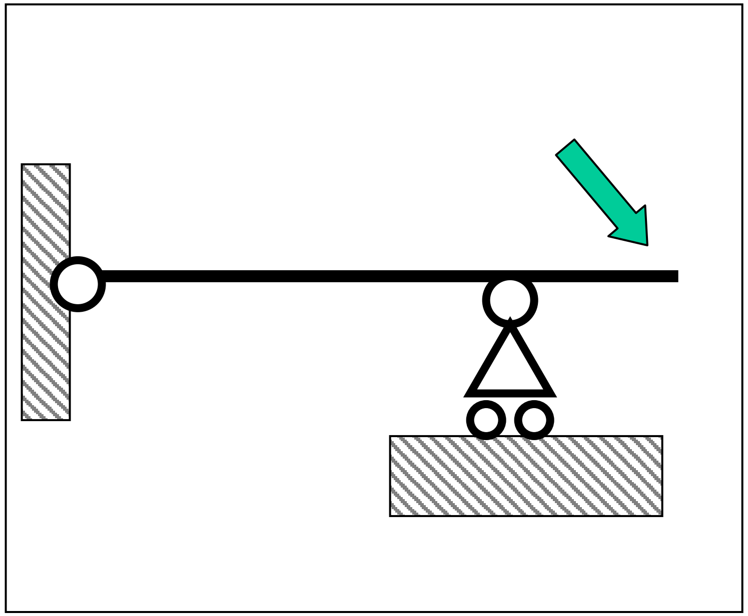
Etc…

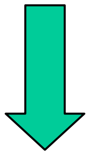
Sistemi di equazioni differenziali alle derivate parziali

$$\begin{cases} \nabla^2 u + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{X}{G} = 0 \\ \nabla^2 v + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{Y}{G} = 0 \\ \nabla^2 w + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{Z}{G} = 0 \end{cases}$$

Soluzioni analitiche: solo in casi particolari, introducendo rilevanti semplificazioni (travi, piastre, gusci…)







Sviluppo di tecniche di soluzione **approssimate**

Il Metodo degli Elementi Finiti (MEF), per la grande versatilità, è di gran lunga il più diffuso.

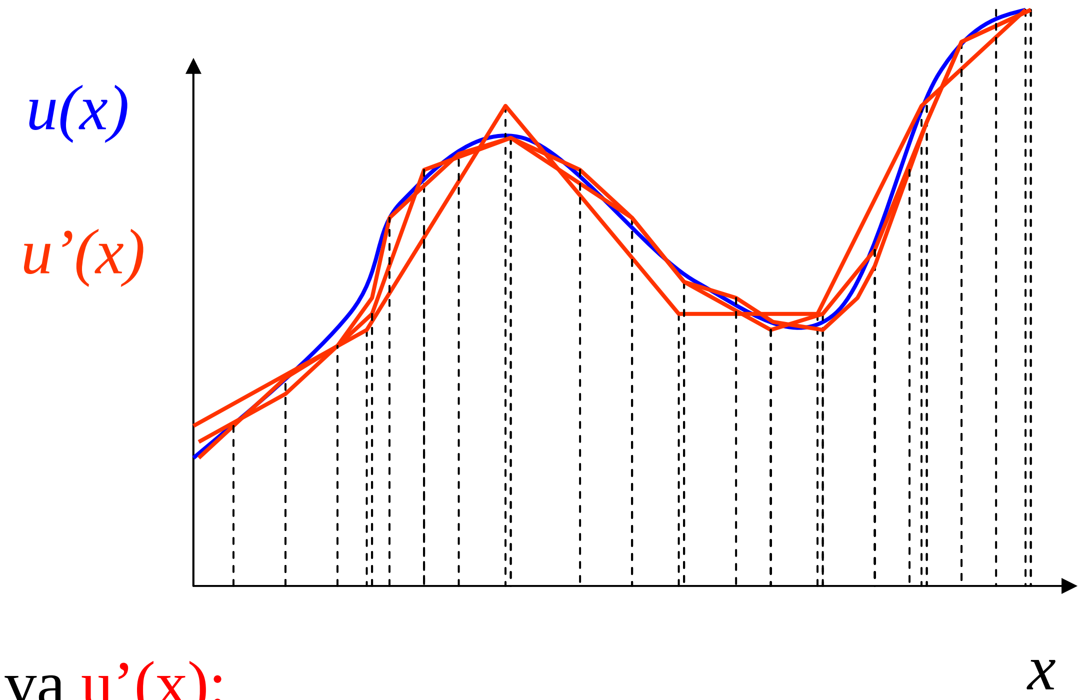
Idea centrale del MEF (e delle altre tecniche approssimate):

Problema originale: determinare le f.ni incognite u, v, w

$$\begin{cases} \nabla^2 u + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{X}{G} = 0 \\ \nabla^2 v + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{Y}{G} = 0 \\ \nabla^2 w + \dfrac{1}{1-2\nu} \cdot \dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{Z}{G} = 0 \end{cases}$$

Problema sostitutivo: determinare delle funzioni sostitutive che approssimino u, v e w con un errore accettabile ai fini pratici e siano relativamente facili da calcolare

Esempio di funzione approssimante
(problema monodimensionale)



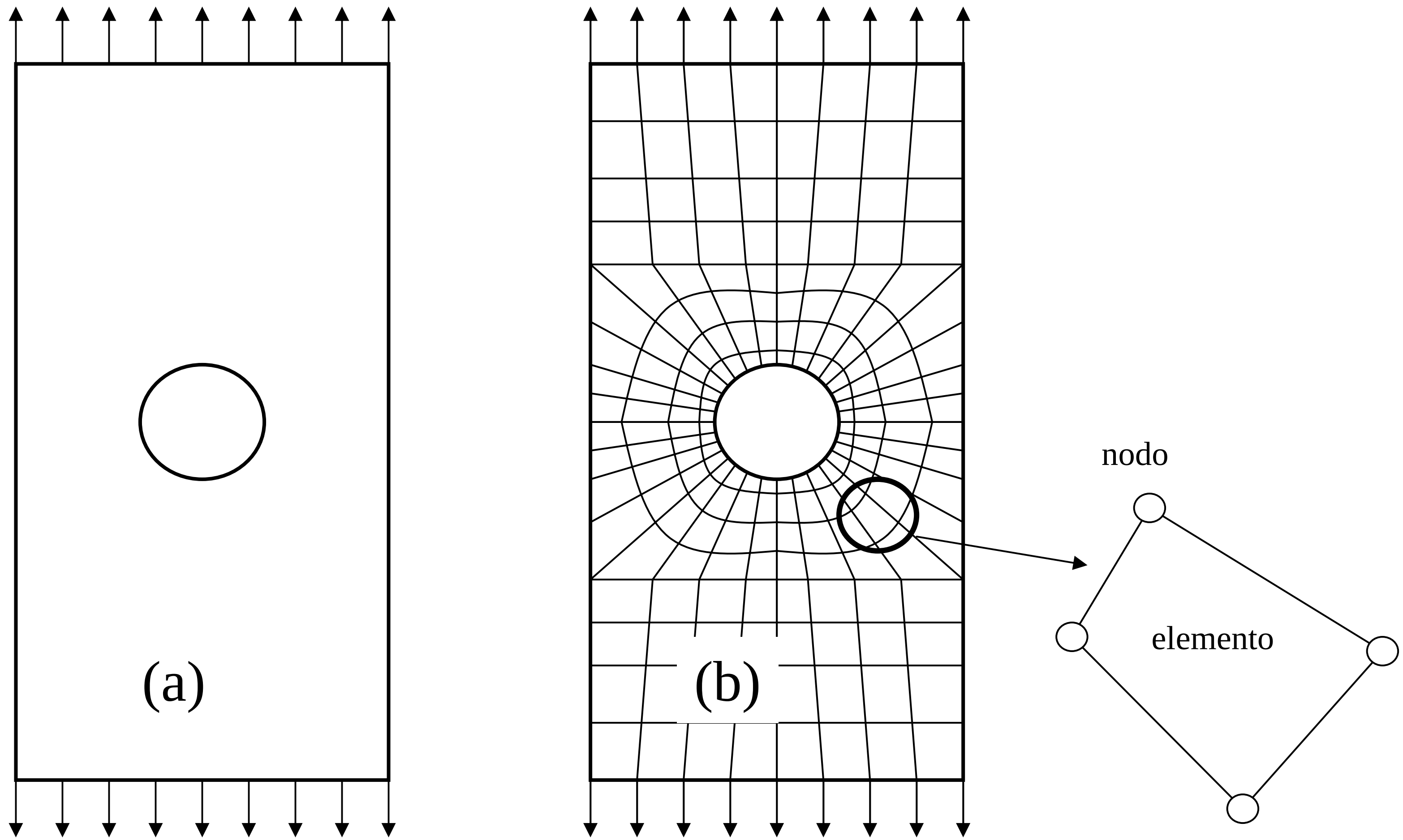
F.ne sostitutiva u'(x):

- espressione matematica semplice
- nota ovunque una volta noto il valore di un n° finito di parametri

Oss.ni:

- necessario assicurare la convergenza
- soluzione affetta da errori

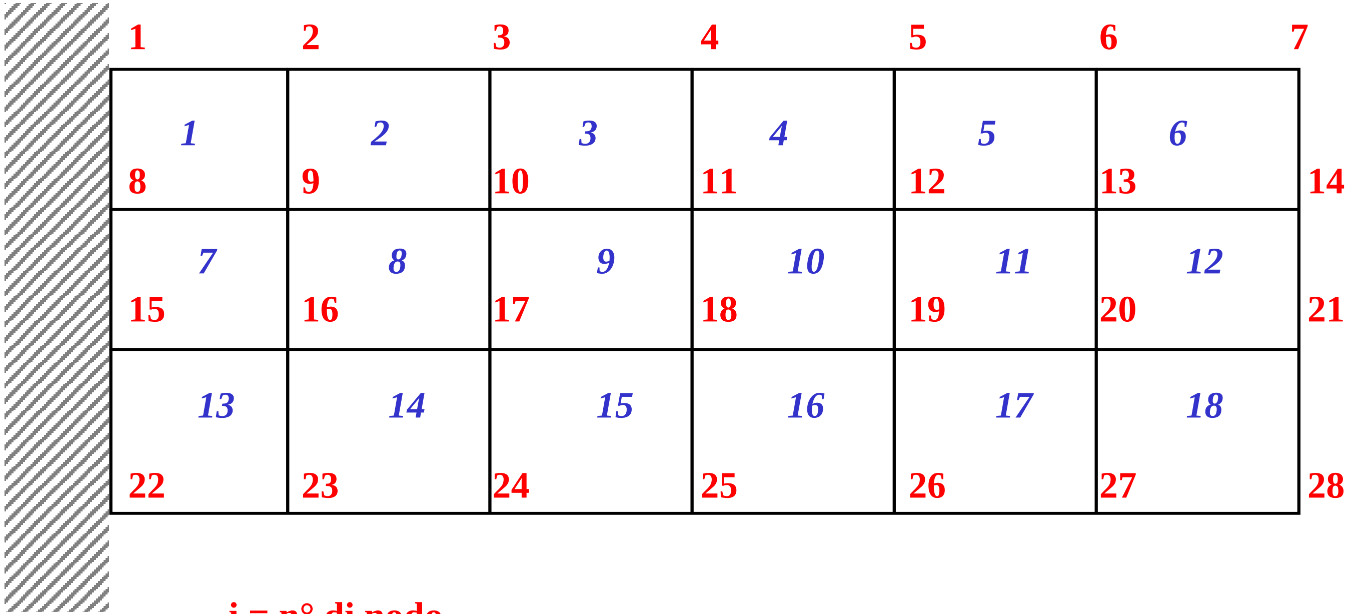
Discretizzazione



Struttura

Modello (“mesh”)

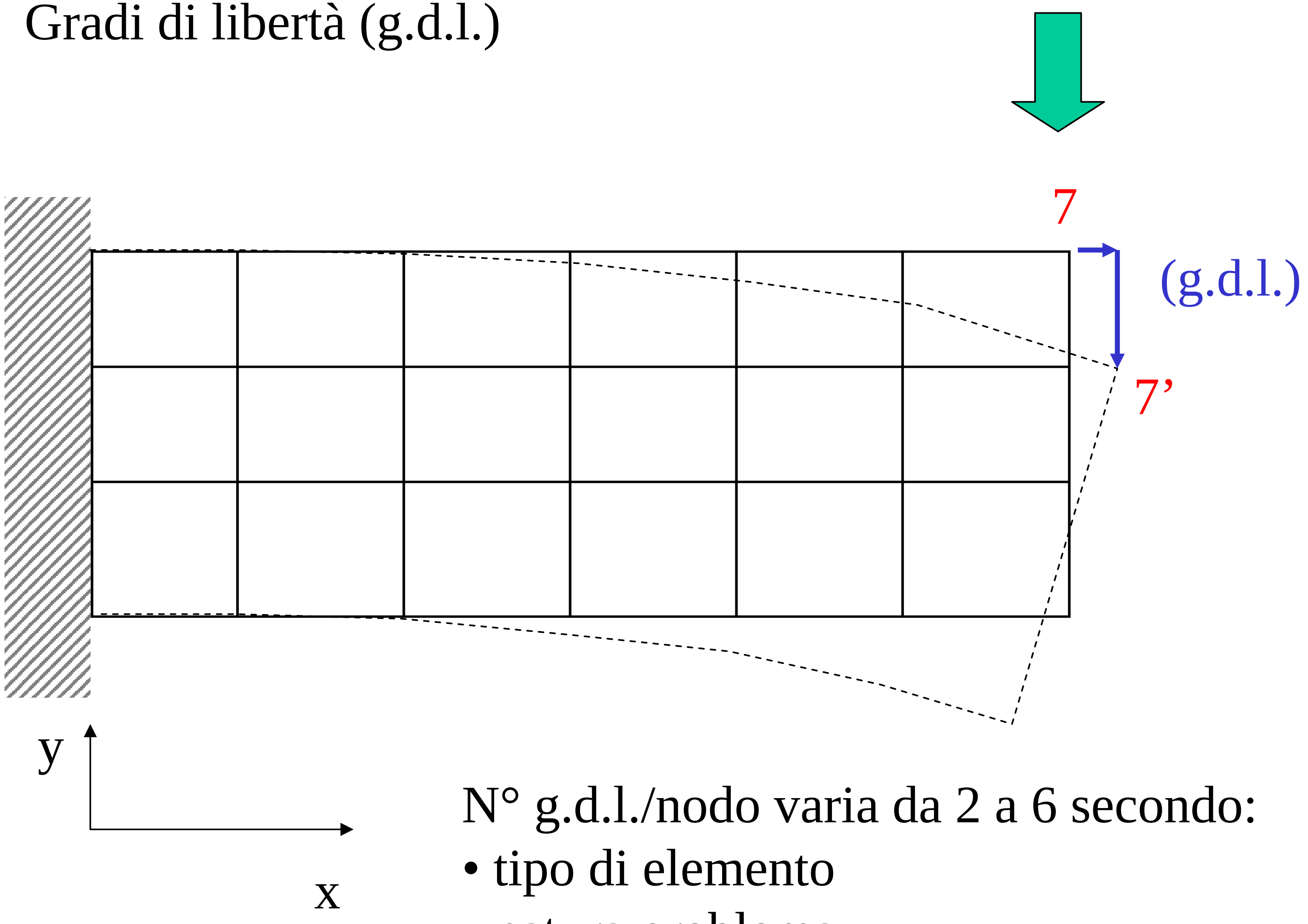
Nodi ed elementi identificati da un numero univoco



i = n° di nodo

***i* = n° di elemento**

Gradi di libertà (g.d.l.)



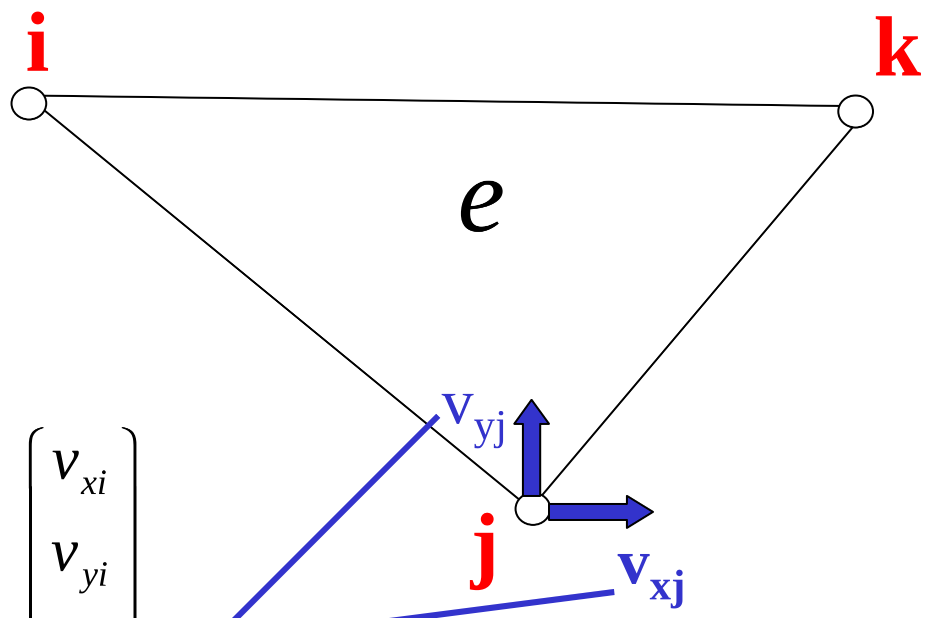
N° g.d.l./nodo varia da 2 a 6 secondo:

- tipo di elemento
- natura problema

N° totale g.d.l. = N° g.d.l./nodo * N° nodi

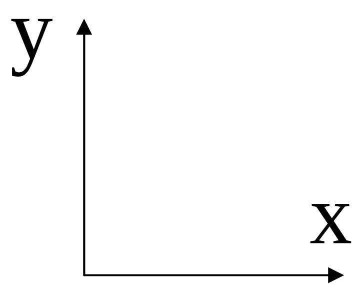
Studio del comportamento meccanico del singolo elemento

Elemento piano per problemi 2D



$$\{U^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix} = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{Bmatrix}$$

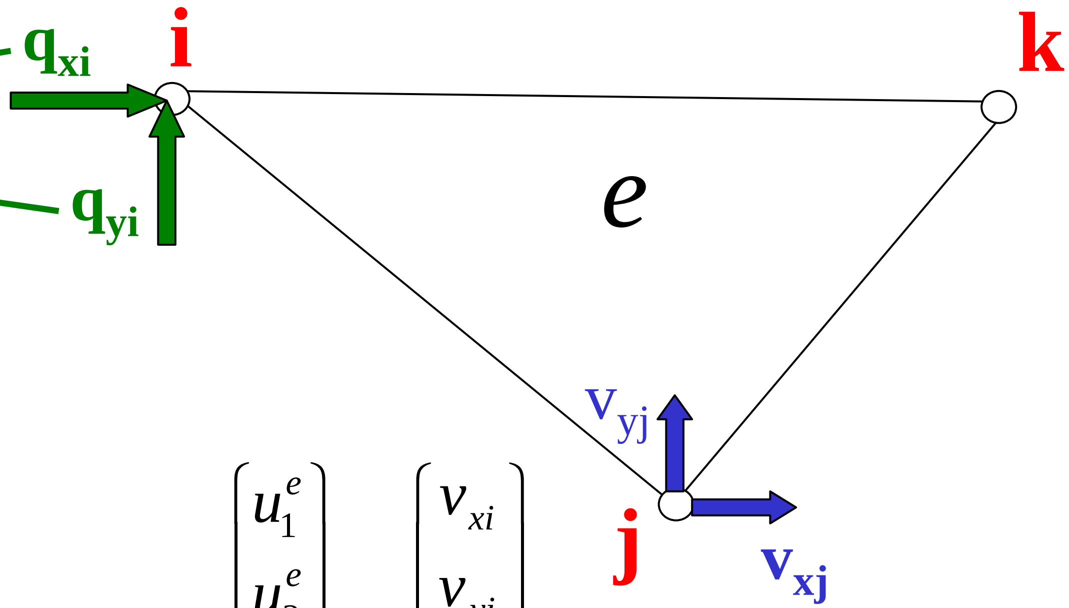
(6 x 1)



Studio del comportamento meccanico del singolo elemento

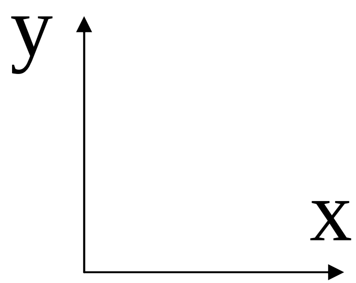
Elemento piano per problemi 2D

$$\{P^e\} = \begin{Bmatrix} p_1^e \\ p_2^e \\ p_3^e \\ p_4^e \\ p_5^e \\ p_6^e \end{Bmatrix} = \begin{Bmatrix} q_{xi} \\ q_{yi} \\ q_{xj} \\ q_{yj} \\ q_{xk} \\ q_{yk} \end{Bmatrix}$$



$$\{U^e\} \xrightarrow{?} \{P^e\}$$

$$\{U^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix} = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{Bmatrix}$$



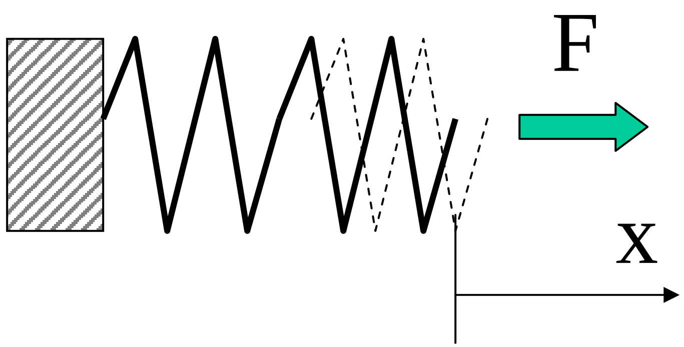
Studio condotto in campo lineare:

$$\{P^e\} = [K^e] \cdot \{U^e\}$$

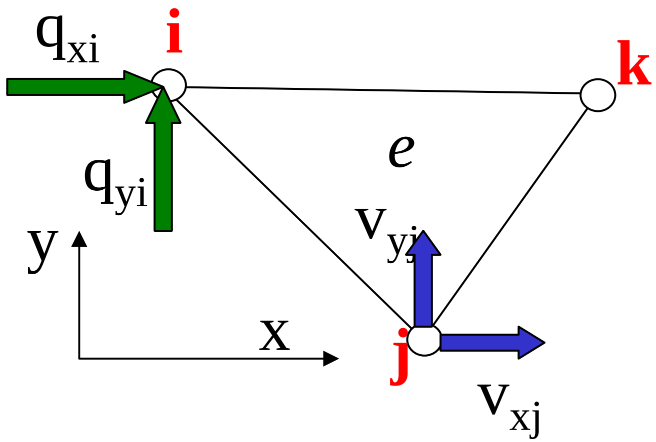
$$6x1 \quad 6x6 \quad 6x1$$

Matrice di rigidezza dell'elemento

Elemento = molla “multidimensionale

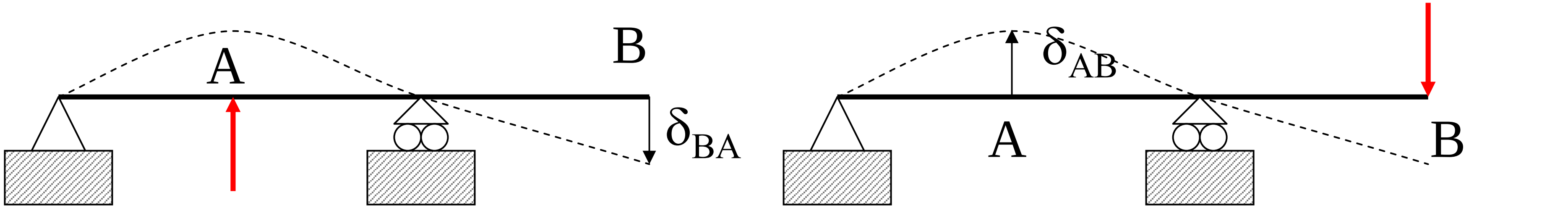


F=k x

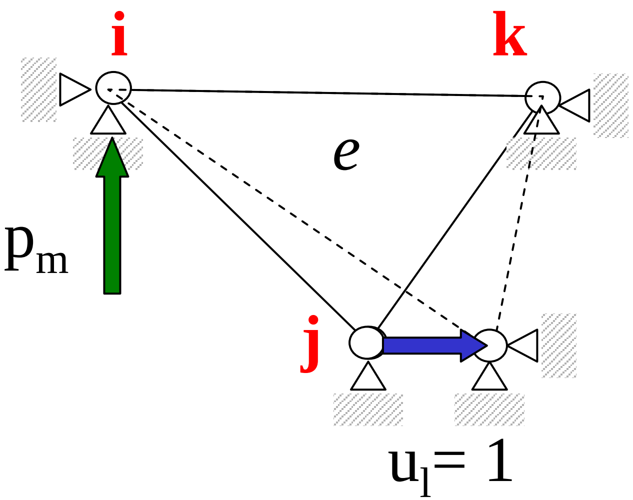


$$\{P^e\} = [K^e] \cdot \{U^e\}$$

Teorema di reciprocità



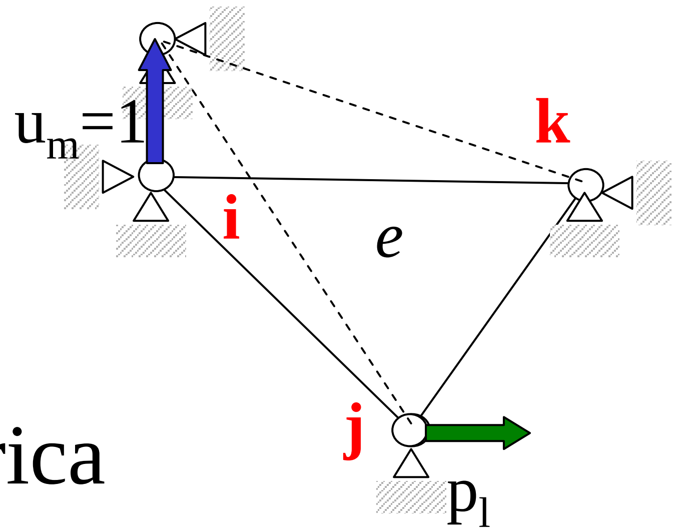
$$\delta_{AB} = \delta_{BA}$$



$$p_m^{\,e} = p_l^{\,e}$$

$$k_{ml} = k_{lm}$$

$[K^e]$ simmetrica

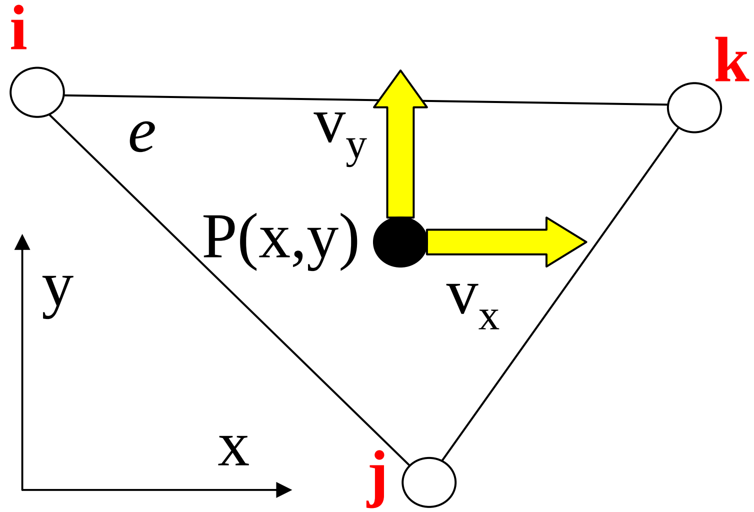


Valutazione di [*Ke*]

Spostamenti nei punti interni all'elemento

$$\{v(x,y)\} = \begin{Bmatrix} v_x(x,y) \\ v_y(x,y) \end{Bmatrix} = \left[N^e(x,y)\right] \cdot \{U^e\}$$

2*x*1 2*x*1 2*x*6 6*x*1

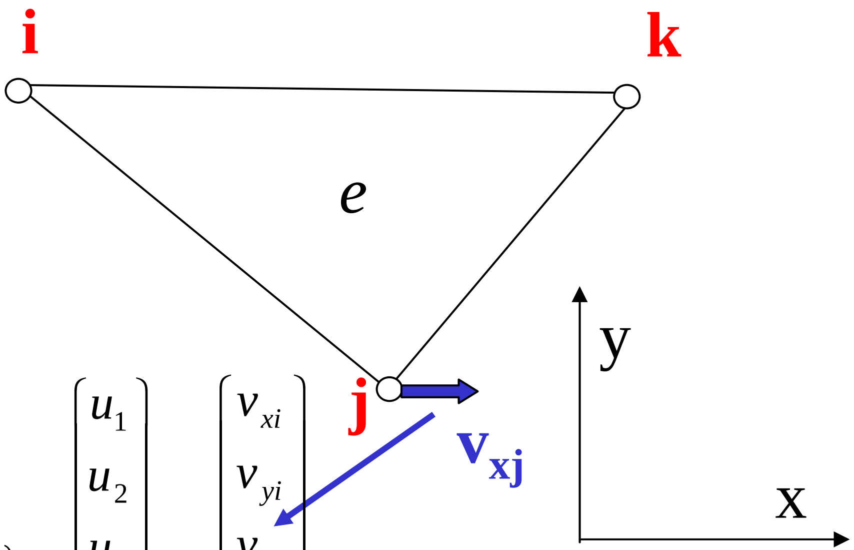


F.ni di forma ("shape functions")

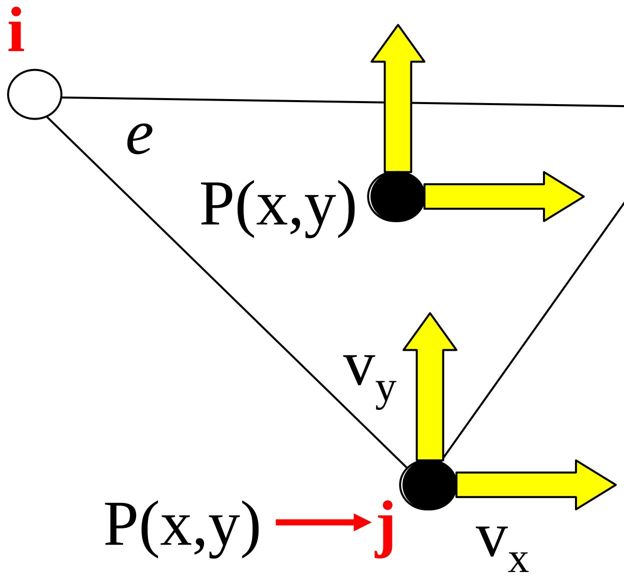
$$v_r = \sum_{l=1}^{6} N^e_{rl}(x,y) \cdot u_l$$

Ogni f.ne di forma rappresenta il "peso" (dipendente dalla posizione di P) che ciascuna componente di spostamento nodale ha nel determinare lo spostamento di P

Pb: - che forma matematica dare alle $N^e(x,y)$?
- come determinare le $N^e(x,y)$?



$$\{U^e\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{Bmatrix}$$



$$v_1(x_j, y_j) = v_x(x_j, y_j) = \sum_{l=1}^{6} N_{1l}^e(x_j, y_j) \cdot u_l =$$

$$= N_{11}^e(x_j, y_j) \cdot u_1 + N_{12}^e(x_j, y_j) \cdot u_2 + ... = u_3$$

$$N_{1l}^e(x_j, y_j) = \begin{cases} 0 \quad se \quad l \neq 3 \\ 1 \quad se \quad l = 3 \end{cases}$$

$$v_1(x_j, y_j) = \sum_{l=1}^{6} N_{1l}^e(x_j, y_j) \cdot u_l = N_{11}^e(x_j, y_j) \cdot u_1 + N_{12}^e(x_j, y_j) \cdot u_2 + \ldots.$$

$$\begin{cases} N_{11}^e(x_i, y_i) = 1 & N_{14}^e(x_i, y_i) = 0 \\ N_{12}^e(x_i, y_i) = 0 & N_{15}^e(x_i, y_i) = 0 \\ N_{13}^e(x_i, y_i) = 0 & N_{16}^e(x_i, y_i) = 0 \end{cases}$$

$$\begin{cases} N_{11}^e(x_j, y_j) = 0 & N_{14}^e(x_j, y_j) = 0 \\ N_{12}^e(x_j, y_j) = 0 & N_{15}^e(x_j, y_j) = 0 \\ N_{13}^e(x_j, y_j) = 1 & N_{16}^e(x_j, y_j) = 0 \end{cases}$$

$$\begin{cases} N_{11}^e(x_k, y_k) = 0 & N_{14}^e(x_k, y_k) = 0 \\ N_{12}^e(x_k, y_k) = 0 & N_{15}^e(x_k, y_k) = 1 \\ N_{13}^e(x_k, y_k) = 0 & N_{16}^e(x_k, y_k) = 0 \end{cases}$$

$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{12}(x_i, y_i) = 0 \\ N_{12}(x_j, y_j) = 0 \\ N_{12}(x_k, y_k) = 0 \end{cases}$$

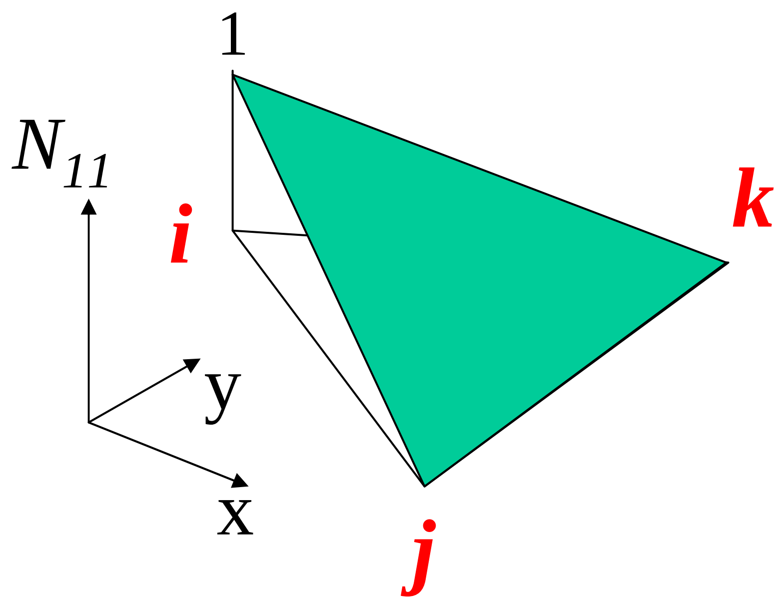
$$\begin{cases} N_{13}(x_i, y_i) = 0 \\ N_{13}(x_j, y_j) = 1 \\ N_{13}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{14}(x_i, y_i) = 0 \\ N_{14}(x_j, y_j) = 0 \\ N_{14}(x_k, y_k) = 0 \end{cases}$$

$$\begin{cases} N_{15}(x_i, y_i) = 0 \\ N_{15}(x_j, y_j) = 0 \\ N_{15}(x_k, y_k) = 1 \end{cases} \quad \begin{cases} N_{16}(x_i, y_i) = 0 \\ N_{16}(x_j, y_j) = 0 \\ N_{16}(x_k, y_k) = 0 \end{cases}$$

$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

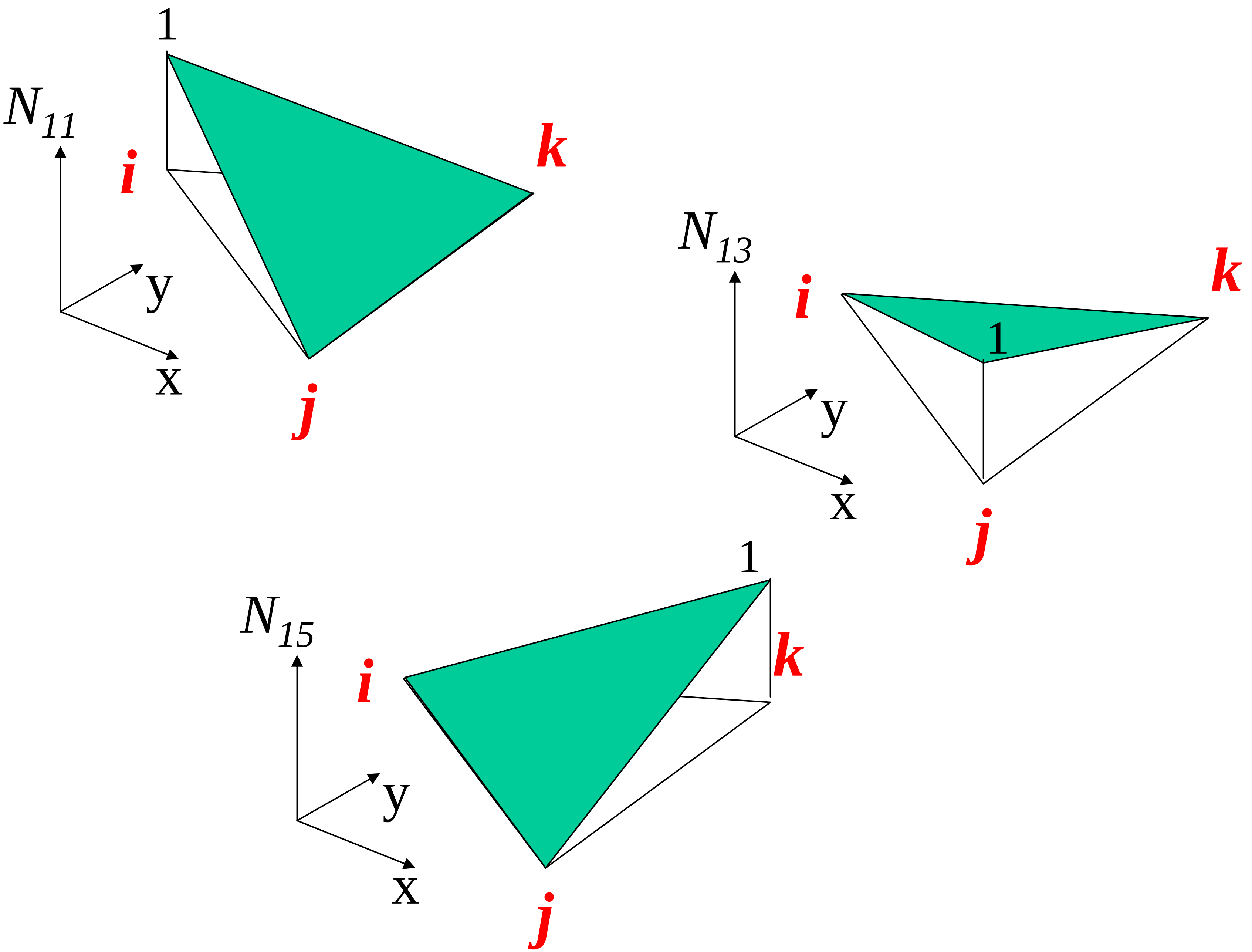
$$N^e_{lm}(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

$$\begin{cases} A_{11} + B_{11}x_i + C_{11}y_i = 1 \\ A_{11} + B_{11}x_j + C_{11}y_j = 0 \\ A_{11} + B_{11}x_k + C_{11}y_k = 0 \end{cases}$$



$$\begin{cases} A_{11} = \dfrac{x_j y_k - x_k y_j}{2\Delta} \\ B_{11} = \dfrac{y_j - y_k}{2\Delta} \\ C_{11} = \dfrac{x_k - x_j}{2\Delta} \end{cases}$$

$$2\Delta = \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

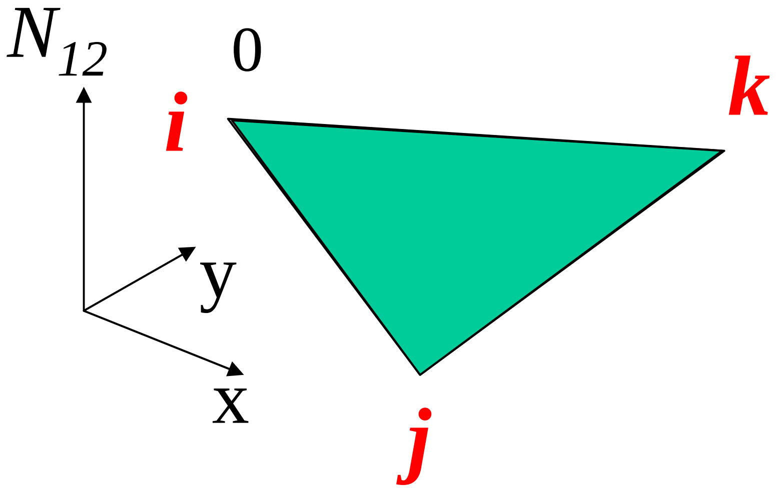


1
N_{11}
i
k
y
x
j
N_{13}
i
k
1
y
x
j
1
N_{15}
i
k
y
x
j

$$\begin{cases} N_{12}(x_i, y_i) = 0 \\ N_{12}(x_j, y_j) = 0 \\ N_{12}(x_k, y_k) = 0 \end{cases}$$

$$N^e_{lm}(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

$$\begin{cases} A_{12} + B_{12}x_i + C_{12}y_i = 0 \\ A_{12} + B_{12}x_j + C_{12}y_j = 0 \\ A_{12} + B_{12}x_k + C_{12}y_k = 0 \end{cases}$$



$$\begin{cases} A_{12} = 0 \\ B_{12} = 0 \\ C_{12} = 0 \end{cases}$$

Matrice delle funzioni di forma

$$\{v(x,y)\} = \begin{Bmatrix} v_x(x,y) \\ v_y(x,y) \end{Bmatrix} = \left[N^e(x,y)\right] \cdot \{U^e\}$$

$$2x1 \qquad\qquad 2x1 \qquad\qquad 2x6 \qquad 6x1$$

$$\begin{bmatrix} N_{11}(x,y) & 0 & N_{13}(x,y) & 0 & N_{15}(x,y) & 0 \\ 0 & N_{22} = N_{11} & 0 & N_{24} = N_{13} & 0 & N_{26} = N_{15} \end{bmatrix}$$

Calcolo delle deformazioni

Spostamenti —*congruenza*→ Deformazioni

$$\begin{cases} \varepsilon_x = \dfrac{\partial v_x}{\partial x} \\ \varepsilon_y = \dfrac{\partial v_y}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x} \end{cases} \longrightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} v_x(x,y) \\ v_y(x,y) \end{Bmatrix} = [L]\{v(x,y)\}$$

$$\{\varepsilon(x,y)\} = [L]\{v(x,y)\}$$

3x1 3x2 2x1

$$\{v(x,y)\} = [N(x,y)]\{U^e\}$$

2x1 2x6 6x1

$$\{\varepsilon\} = [L][N]\{U^e\} = [B]\{U^e\}$$

3x1 3x6 6x1

Contenuto matrice [B]

$$[B]=[L][N]=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} N_{11} & 0 & N_{13} & 0 & N_{15} & 0 \\ 0 & N_{22} & 0 & N_{24} & 0 & N_{26} \end{bmatrix}$$

$$[B]=\begin{bmatrix} \frac{\partial N_{11}}{\partial x} & 0 & \frac{\partial N_{13}}{\partial x} & 0 & \frac{\partial N_{15}}{\partial x} & 0 \\ 0 & \frac{\partial N_{22}}{\partial y} & 0 & \frac{\partial N_{24}}{\partial y} & 0 & \frac{\partial N_{26}}{\partial y} \\ \frac{\partial N_{11}}{\partial y} & \frac{\partial N_{22}}{\partial x} & \frac{\partial N_{13}}{\partial y} & \frac{\partial N_{24}}{\partial x} & \frac{\partial N_{15}}{\partial y} & \frac{\partial N_{26}}{\partial x} \end{bmatrix}$$

Relazioni costitutive

Esempio 1: stato piano di tensione, materiale isotropo

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu\sigma_y}{E} \\ \varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu\sigma_x}{E} \\ \gamma_{xy} = \dfrac{2(1+\nu)\tau_{xy}}{E} \end{cases}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

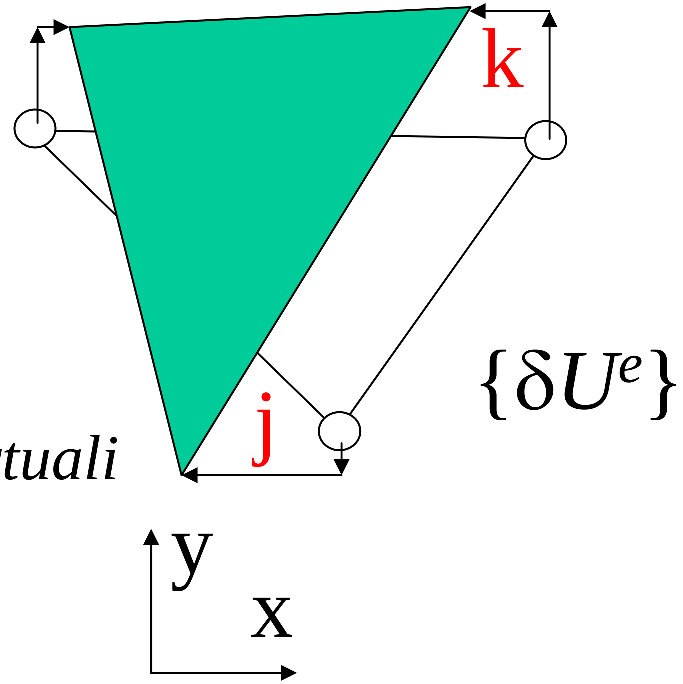
Valutazione di $[K^e]$

Principio dei Lavori Virtuali

$$L_{est} = L_{int}$$

*Carichi nodali veri * spost.nodali virtuali*

*Tensioni vere * deformazioni virtuali*



$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

Spost. virtuali

Carichi effettivi

$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$

$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D]\{\varepsilon\} dV$$

$$\{\varepsilon\} = [B]\{U^e\}$$

$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D][B]\{U^e\} dV = \{\delta U^e\}^T \int_V [B]^T [D][B] dV \{U^e\}$$

$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

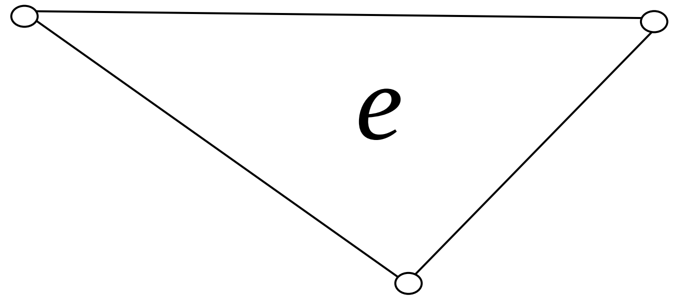
$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D][B] dV \; \{U^e\}$$

$$\{\delta U^e\}^T \{P^e\} = \{\delta U^e\}^T \int_V [B]^T [D][B] dV \; \{U^e\}$$

$$\{P^e\} = \int_V [B]^T [D][B] dV \; \{U^e\}$$

$$\{P^e\} = [K^e] \; \{U^e\}$$

Applicazione



$$\left[K^e\right] = \int_V \left[B\right]^T \left[D\right]\left[B\right] dV$$

$$\left[B\right] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$

$$\left[D\right] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \left(1-\nu\right)/2 \end{bmatrix}$$

$$\left[K^e\right] = \left[B\right]^T \left[D\right]\left[B\right] \int_V dV = \left[B\right]^T \left[D\right]\left[B\right] V$$

Osservazione: unità di misura

$$\left[K^e\right] = \left[B\right]^T \left[D\right] \left[B\right] V$$

$N\,m^{-1}$ → $\left[K^e\right]$

m^{-1} → $\left[B\right]^T$

$N\,m^{-2}$ → $\left[D\right]$

m^{-1} → $\left[B\right]$

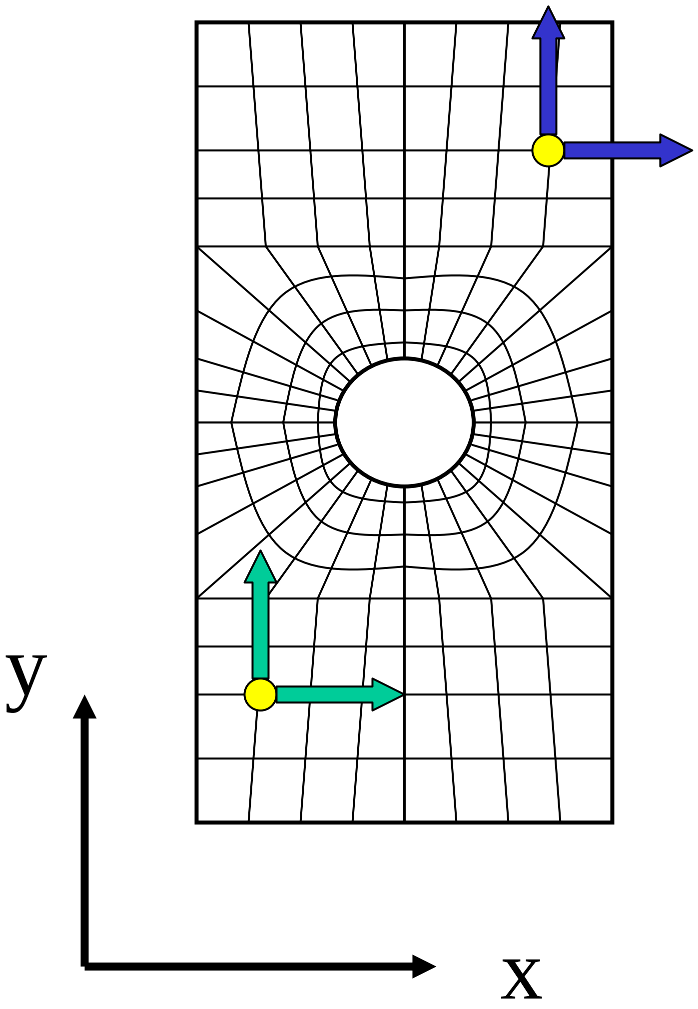
m^3 → V

ANALISI INTERA STRUTTURA

Congruenza → [B]

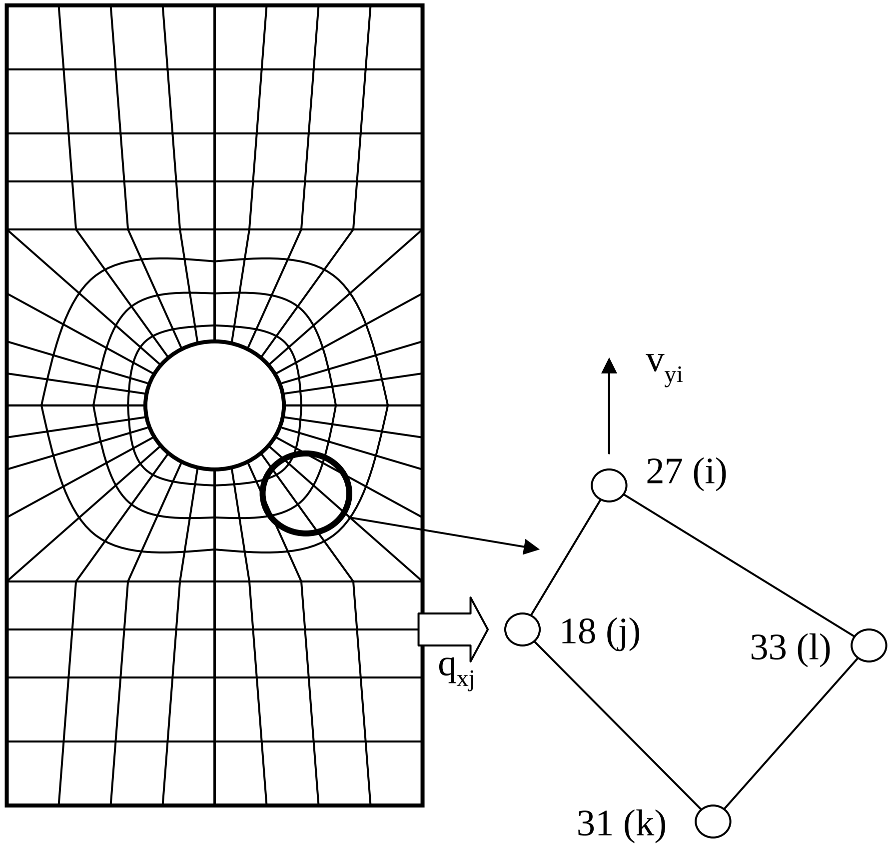
Costitutive → [D]

Equilibrio → Garantito per il singolo elemento (non ancora per la struttura)

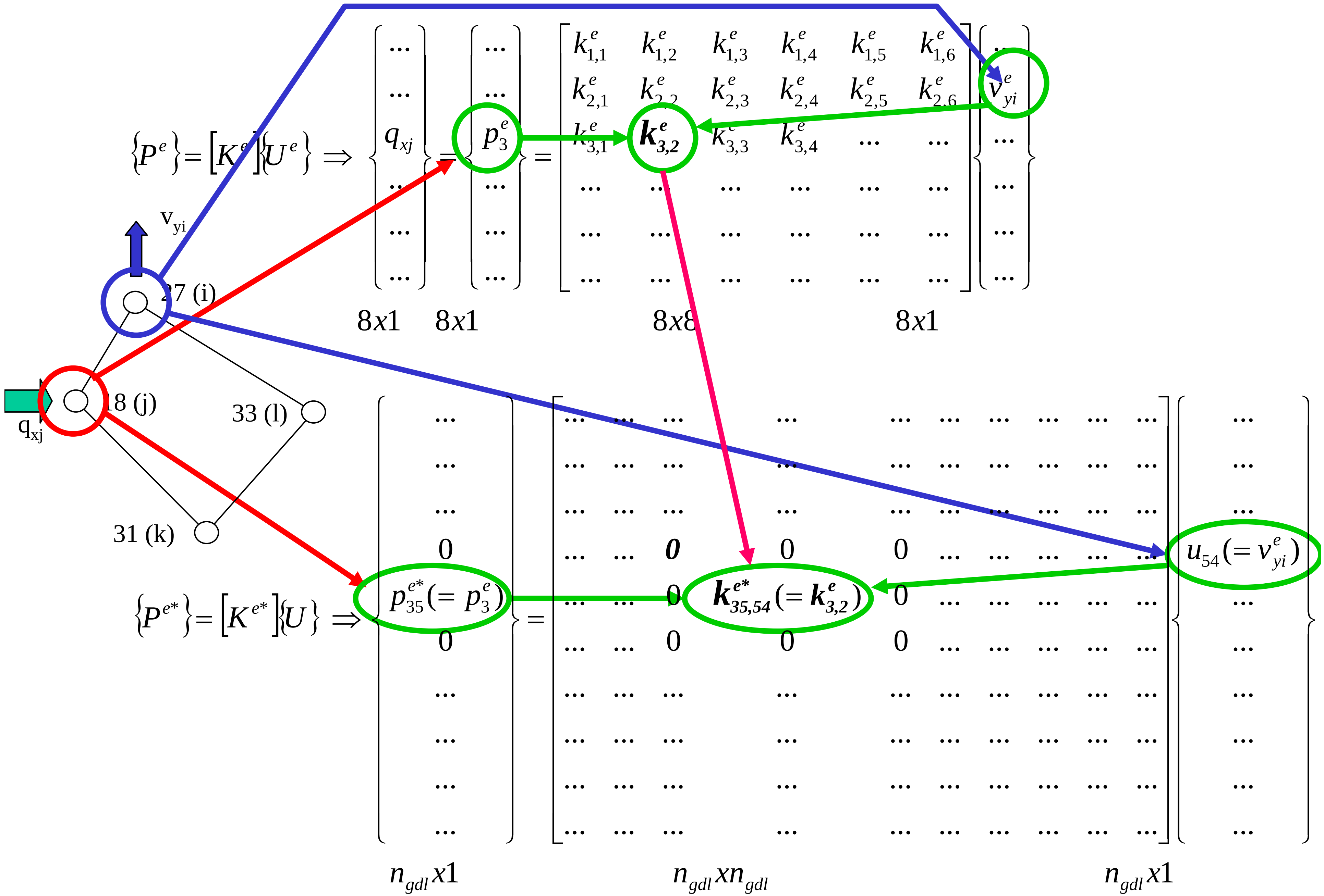


$$\{U\} = \left\{\begin{matrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ - \\ - \\ - \\ v_{yn_N} \end{matrix}\right\} = \left\{\begin{matrix} u_1 \\ u_2 \\ u_3 \\ - \\ - \\ - \\ u_{n_{GDL}} \end{matrix}\right\}$$

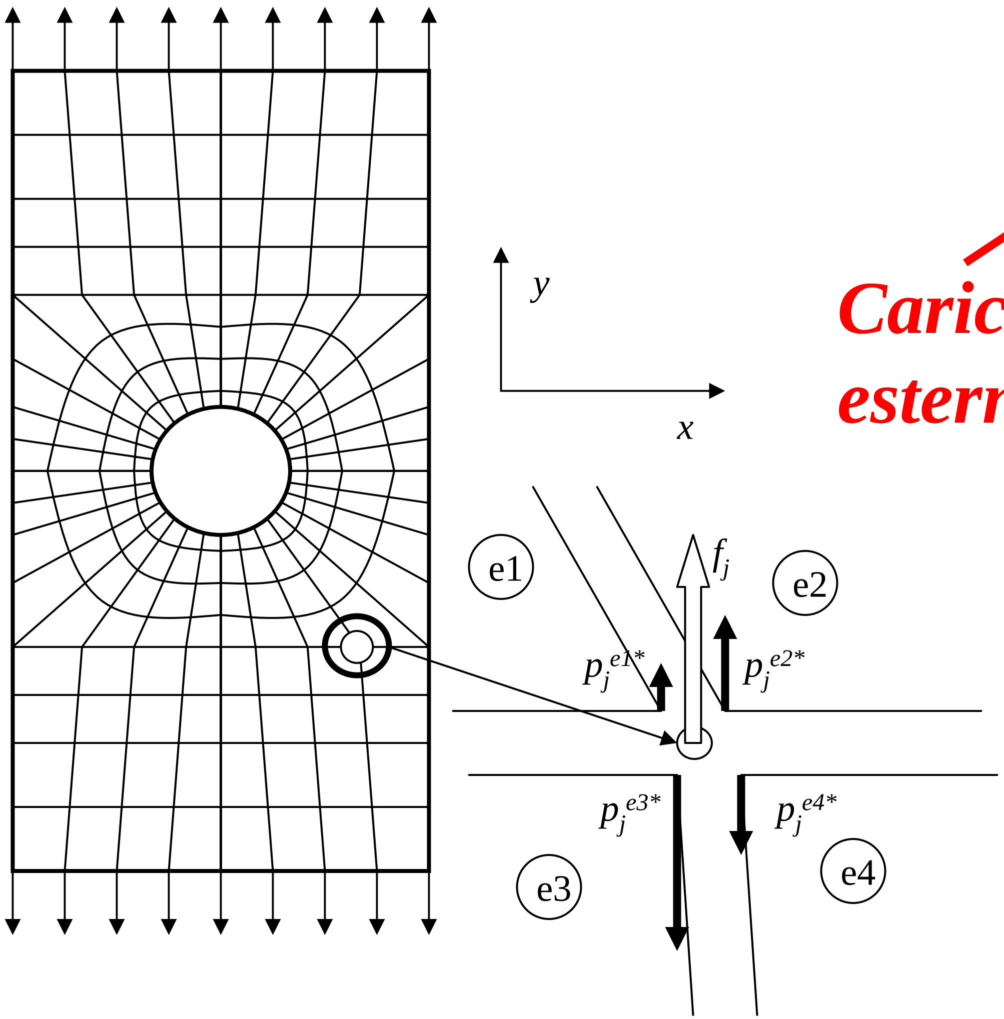
$$\{F\} = \left\{\begin{matrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ - \\ - \\ - \\ f_{yn_N} \end{matrix}\right\} = \left\{\begin{matrix} f_1 \\ f_2 \\ f_3 \\ - \\ - \\ - \\ f_{n_{GDL}} \end{matrix}\right\}$$



v_{yi}
27 (i)
18 (j)
33 (l)
31 (k)
q_{xj}



$\{P^e\}=[K^e]\{U^e\} \Rightarrow$
q_{xj}
p_3^e
$k_{1,1}^e$ $k_{1,2}^e$ $k_{1,3}^e$ $k_{1,4}^e$ $k_{1,5}^e$ $k_{1,6}^e$
$k_{2,1}^e$ $k_{2,2}^e$ $k_{2,3}^e$ $k_{2,4}^e$ $k_{2,5}^e$ $k_{2,6}^e$
$k_{3,1}^e$ $\boldsymbol{k_{3,2}^e}$ $k_{3,3}^e$ $k_{3,4}^e$
v_{yi}^e
8x1
8x1
8x8
8x1
v_{yi}
27 (i)
18 (j)
33 (l)
31 (k)
q_{xj}
$\{P^{e*}\}=[K^{e*}]\{U\} \Rightarrow$
$p_{35}^{e*}(=p_3^e)$
$\boldsymbol{k_{35,54}^{e*}}(=\boldsymbol{k_{3,2}^e})$
$u_{54}(=v_{yi}^e)$
$n_{gdl}x1$
$n_{gdl}xn_{gdl}$
$n_{gdl}x1$



$$f_j - \sum_{e=1}^{n_E} p_j^{e*} = 0$$

Carico esterno

Carico applicato nel nodo all'elemento "e"

$$f_j = \sum_{e=1}^{n_E} p_j^{e*}$$

$$\{P^{e*}\} = [K^{e*}]\{U\}$$

$$f_j = \sum_{e=1}^{n_E} p_j^{e*} = \sum_{e=1}^{n_E}\left(\sum_{i=1}^{n_{gdl}} k_{ji}^{e*} u_i\right) = \sum_{i=1}^{n_{gdl}}\left(\sum_{e=1}^{n_E} k_{ji}^{e*}\right) u_i$$

Matrice di rigidezza della struttura

$$\{F\} = [K]\{U\}$$

$\{F\}$: $n_{GDL} \times 1$

$[K]$: $n_{GDL} \times n_{GDL}$

$\{U\}$: $n_{GDL} \times 1$

$$k_{ji} = \sum_{e=1}^{n_E} k_{ji}^{e*}$$

SOLUZIONE

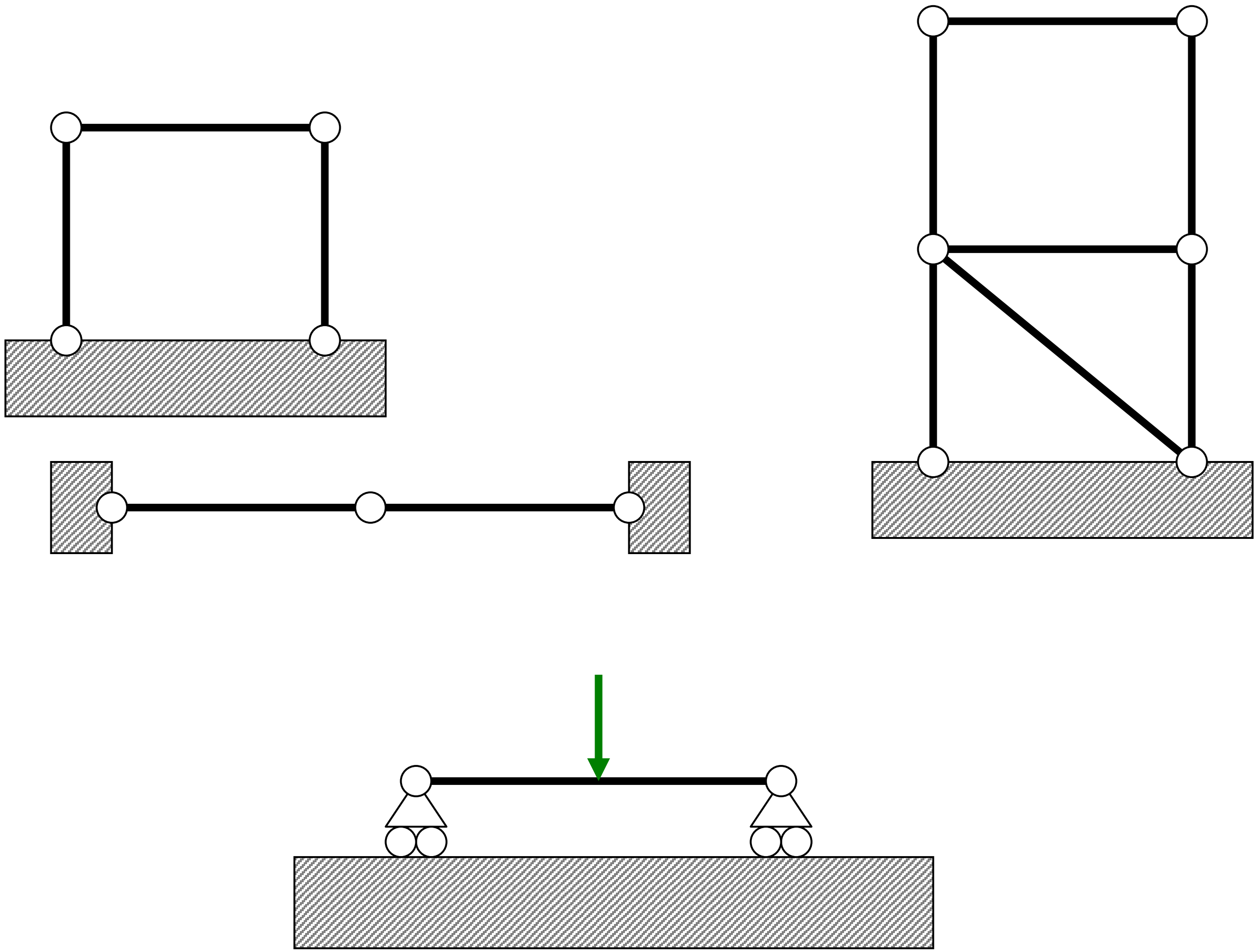
$$\{F\} = [K]\{U\}$$

⇓

$$\{U\} = [K]^{-1}\{F\}$$

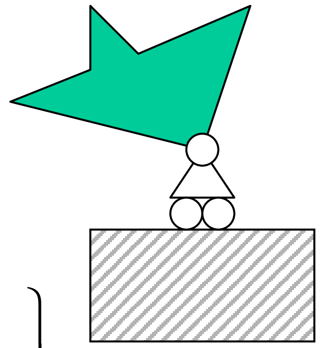
c.n.s. : $\det[K] \neq 0$

$$\det[K] \neq 0 \quad \Rightarrow \quad \text{Struttura non labile}$$



VINCOLI

Vincolare = assegnare "a priori" il valore di una delle componenti di spostamento (g.d.l.)

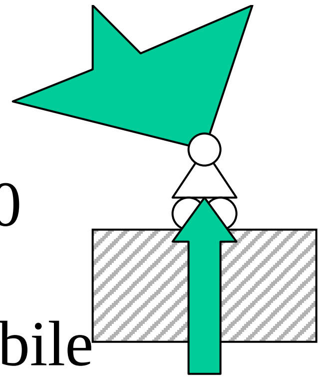


$u_m=0$

$$
\begin{Bmatrix} f_1 \\ f_2 \\ - \\ - \\ f_m \\ - \\ f_{n_{GDL}} \end{Bmatrix}
=
\begin{bmatrix}
k_{11} & k_{12} & - & - & k_{1m} & - & k_{1n_{GDL}} \\
k_{21} & k_{22} & - & - & k_{2m} & - & k_{2n_{GDL}} \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & - \\
k_{m1} & k_{m2} & - & - & k_{m,m} & - & k_{mn_{GDL}} \\
- & - & - & - & - & - & - \\
k_{n_{GDL}1} & k_{n_{GDL}2} & - & - & k_{n_{GDL}m} & - & k_{n_{GDL}n_{GDL}}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ - \\ - \\ u_m \\ - \\ u_{n_{GDL}} \end{Bmatrix}
$$

$n_{GDL} \bullet 1$ $\qquad$ $n_{GDL} \bullet n_{GDL}$ $\qquad$ $n_{GDL} \bullet 1$

$u_m=0$



f_m non assegnabile

$$
\begin{Bmatrix} f_1 \\ f_2 \\ - \\ - \\ f_m \\ - \\ f_{n_{GDL}} \end{Bmatrix}
- u_m \begin{Bmatrix} k_{1m} \\ k_{2m} \\ - \\ - \\ k_{m,m} \\ - \\ k_{n_{GDL}m} \end{Bmatrix}
=
\begin{bmatrix}
k_{11} & k_{12} & - & k_{1m-1} & k_{1m+1} & - & k_{1n_{GDL}} \\
k_{21} & k_{22} & - & k_{2m-1} & k_{2m+1} & - & k_{2n_{GDL}} \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & - \\
k_{m1} & k_{m2} & - & k_{m,m-1} & k_{m,m+1} & - & k_{mn_{GDL}} \\
- & - & - & - & - & - & - \\
k_{n_{GDL}1} & k_{n_{GDL}2} & - & k_{n_{GDL}m-1} & k_{n_{GDL}m+1} & - & k_{n_{GDL}n_{GDL}}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ - \\ u_{m-1} \\ u_{m+1} \\ - \\ u_{n_{GDL}} \end{Bmatrix}
$$

$n_{GDL} \cdot 1$ $\qquad$ $n_{GDL} \cdot (n_{GDL}-1)$ $\qquad$ $(n_{GDL}-1) \cdot 1$

Introduzione vincolo = riduzione di 1 del numero di incognite ed equazioni

$$
\begin{Bmatrix} f_1 \\ f_2 \\ - \\ f_{m-1} \\ f_{m+1} \\ - \\ f_{n_{GDL}} \end{Bmatrix} - u_m \begin{Bmatrix} k_{1m} \\ k_{2m} \\ - \\ k_{m-1.m} \\ k_{m+1,m} \\ - \\ k_{n_{GDL}m} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & - & k_{1m-1} & k_{1m+1} & - & k_{1n_{GDL}} \\ k_{21} & k_{22} & - & k_{2m-1} & k_{2m+1} & - & k_{2n_{GDL}} \\ - & - & - & - & - & - & - \\ k_{m-1,1} & k_{m-1,1} & - & k_{m-1,m-1} & k_{m-1,m+1} & - & k_{m-1,n_{GDL}} \\ k_{m+1,1} & k_{m+1,2} & - & k_{m+1,m-1} & k_{m+1,m+1} & - & k_{m+1,n_{GDL}} \\ - & - & - & - & - & - & - \\ k_{n_{GDL}1} & k_{n_{GDL}2} & - & k_{n_{GDL}m-1} & k_{n_{GDL}m+1} & - & k_{n_{GDL}n_{GDL}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ - \\ u_{m-1} \\ u_{m+1} \\ - \\ u_{n_{GDL}} \end{Bmatrix}
$$

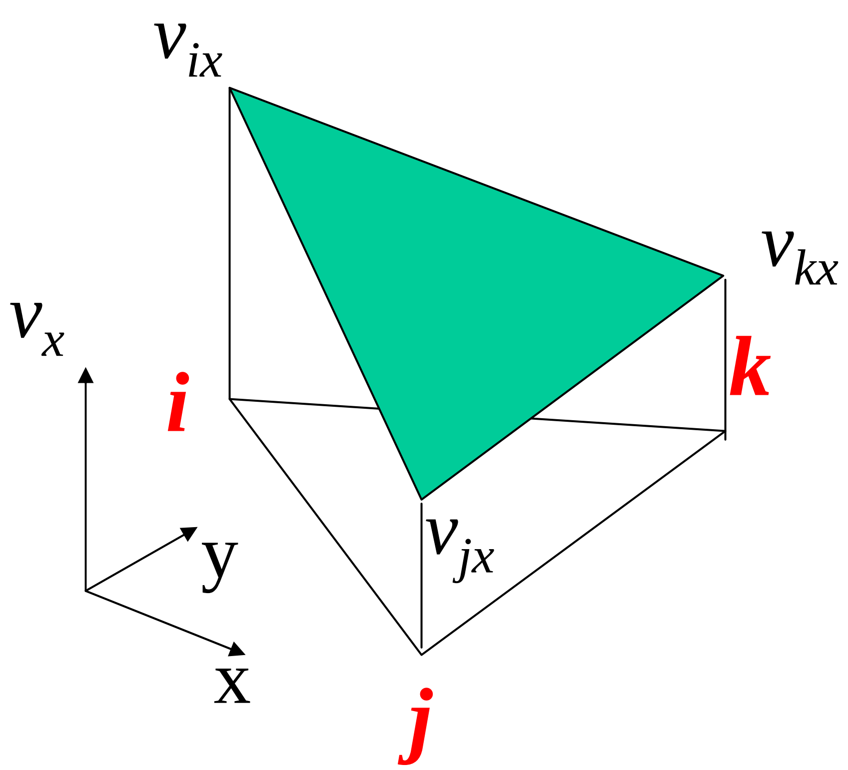
$(n_{GDL}\text{-}1) \bullet 1$ $(n_{GDL}\text{-}1) \bullet (n_{GDL}\text{-}1)$ $(n_{GDL}\text{-}1) \bullet 1$

$$
[K] = \begin{bmatrix}
X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & X & 0 & X & X & 0 & 0 & 0 & 0 & 0 \\
 & & & & X & X & X & X & 0 & 0 & 0 & 0 \\
 & & & & & X & X & 0 & X & 0 & 0 & 0 \\
 & & & & & & X & X & X & 0 & 0 & 0 \\
 & & & & & & & X & X & X & X & 0 \\
 & S & I & M & M & . & & & X & X & X & X \\
 & & & & & & & & & X & X & X \\
 & & & & & & & & & & X & X \\
 & & & & & & & & & & & X
\end{bmatrix}
$$

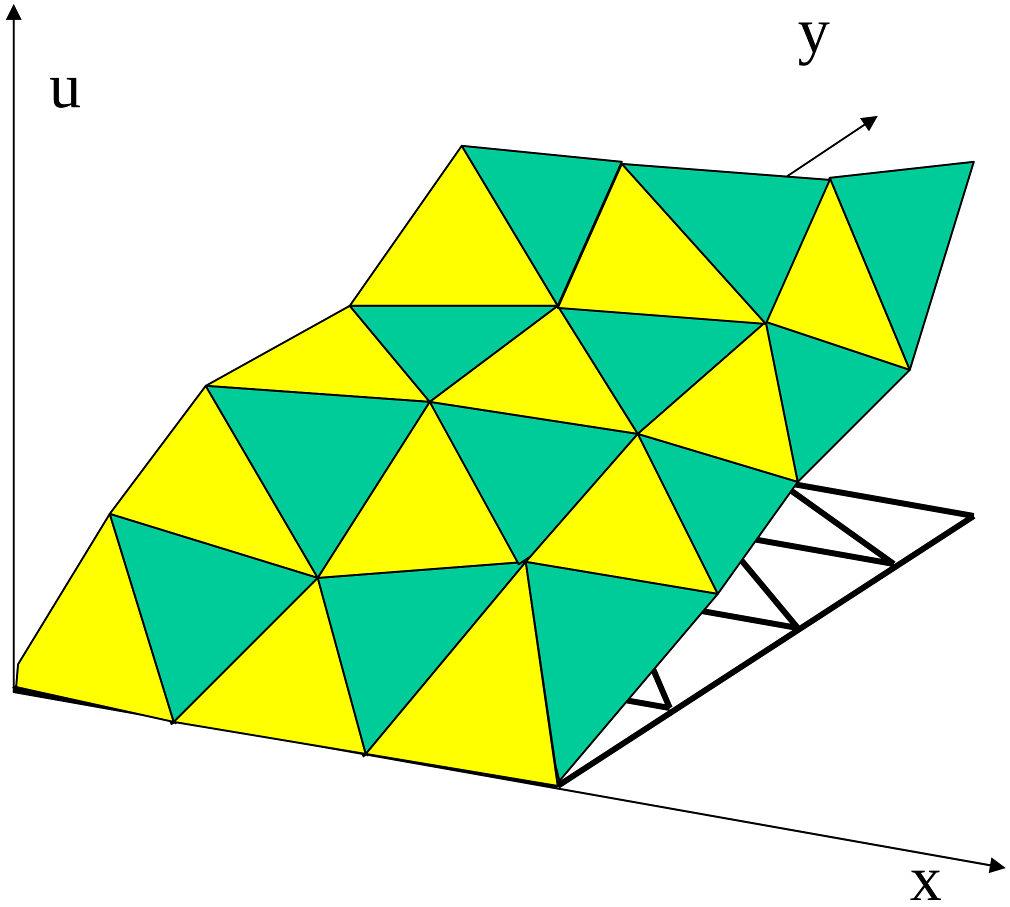
La matrice [K]:

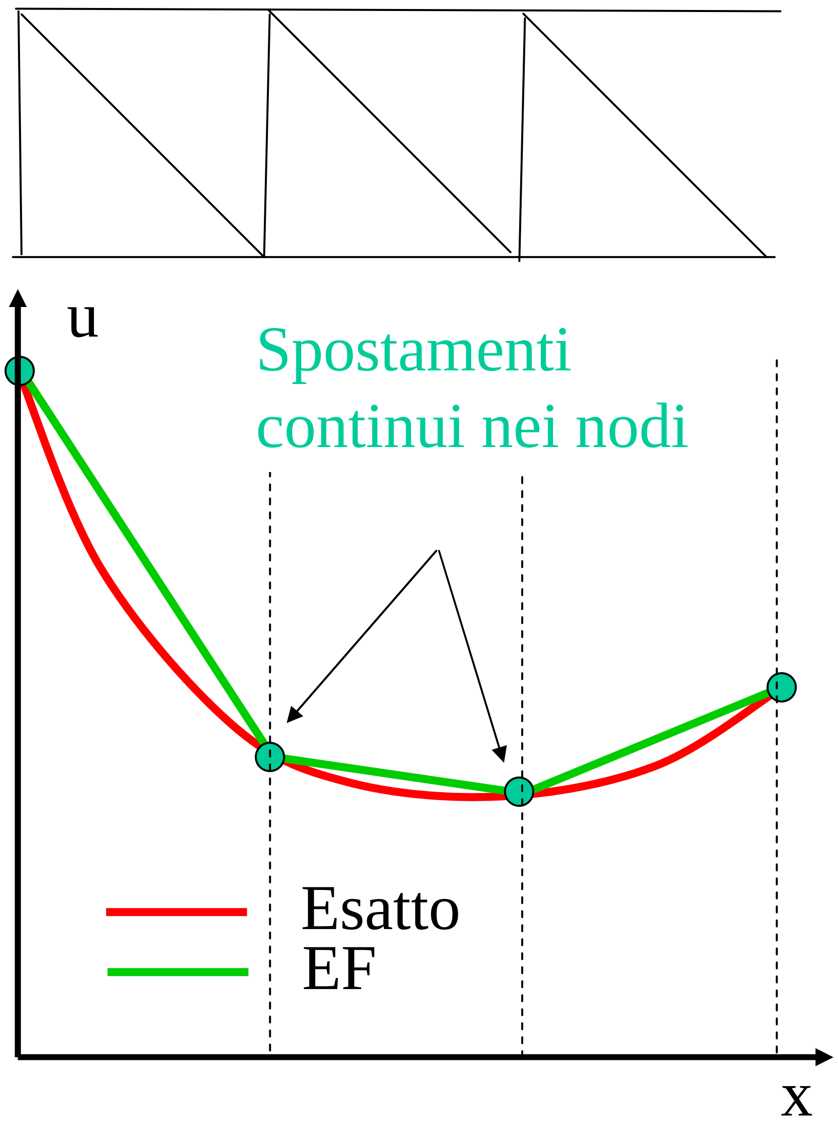
- è simmetrica
- ha una struttura "a banda" attorno alla diagonale principale

Approssimazione effettiva del campo di spostamenti sul singolo elemento



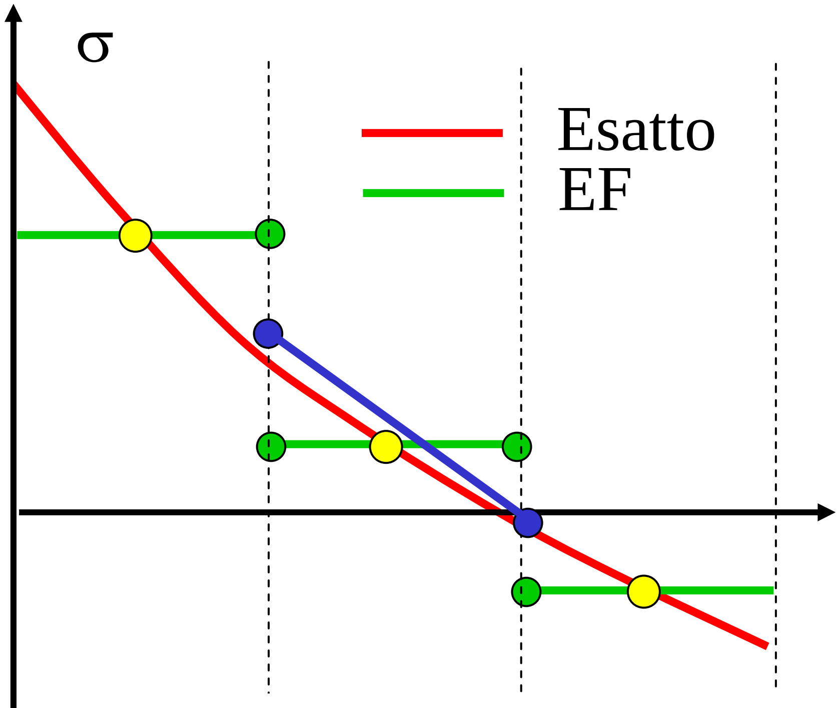
Approssimazione effettiva del campo di spostamenti sull'intero modello





Andamento effettivo delle tensioni

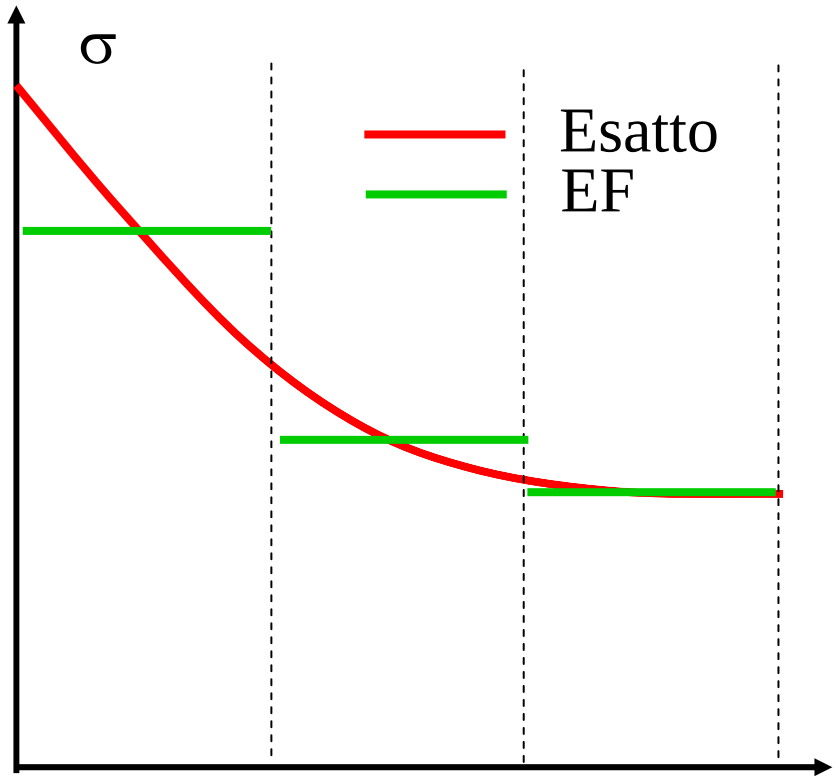
Tensioni discontinue nei nodi



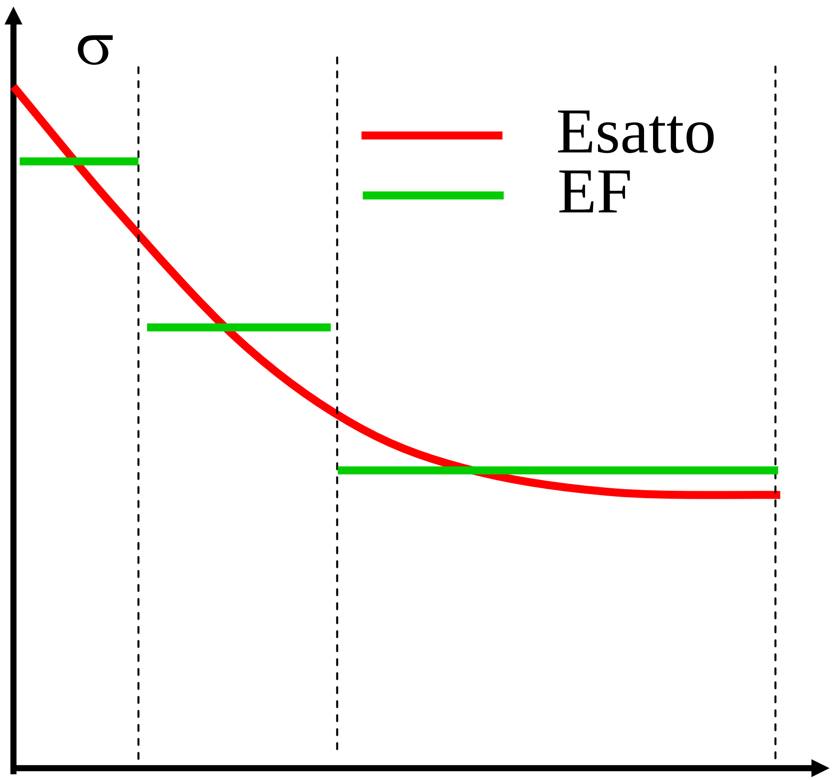
Calcolo di valori mediati nei nodi
(media aritmetica o altre tecniche)

Interpolazione dei valori mediati nodali
nelle zone interne (Es. tramite le N)

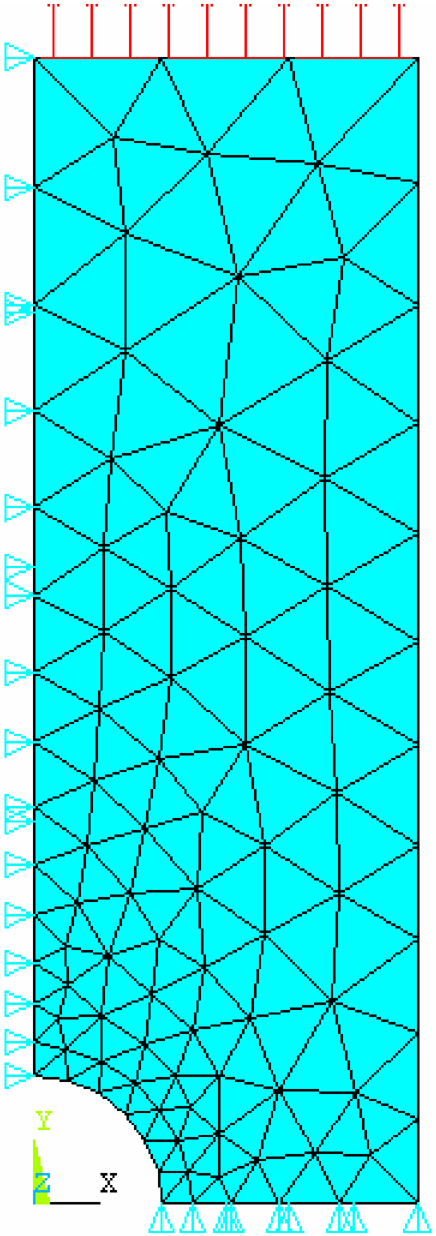
Dimensioni ottimali degli elementi



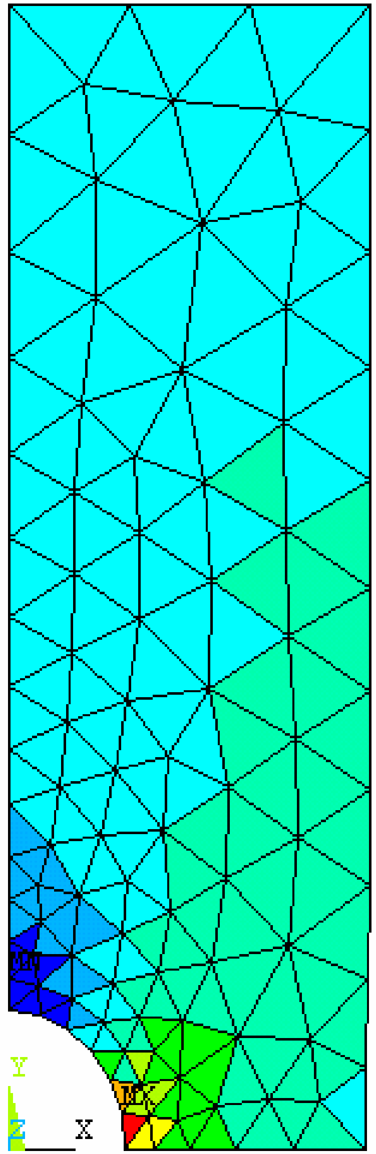
Dimensioni elementi
non ottimali



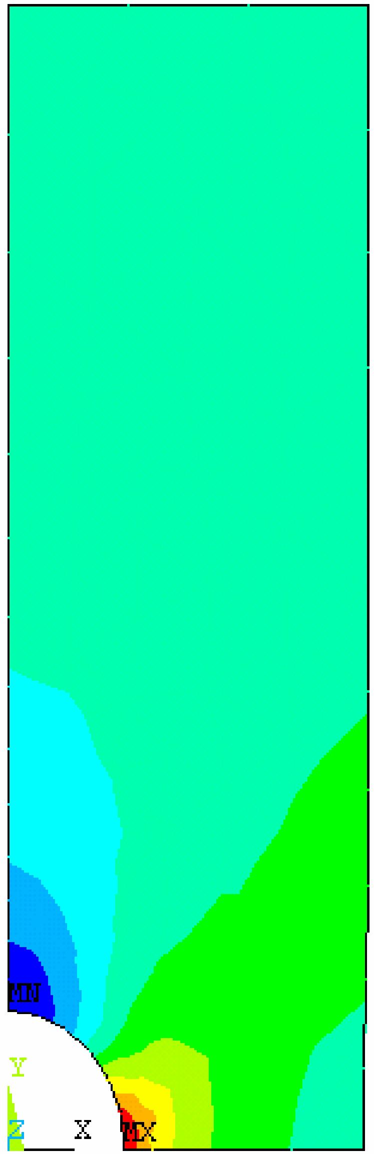
Dimensioni elementi
ottimali



Modello

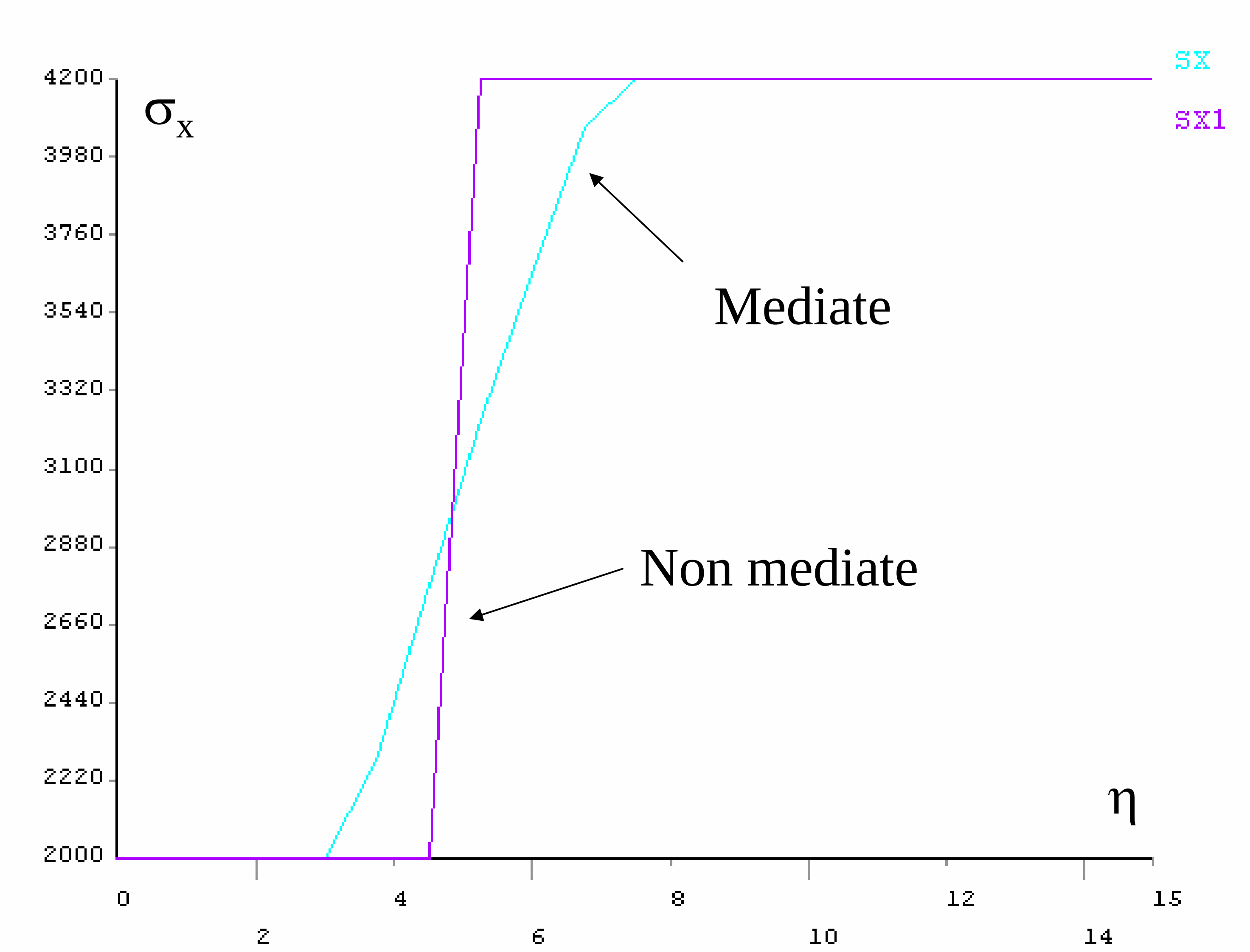


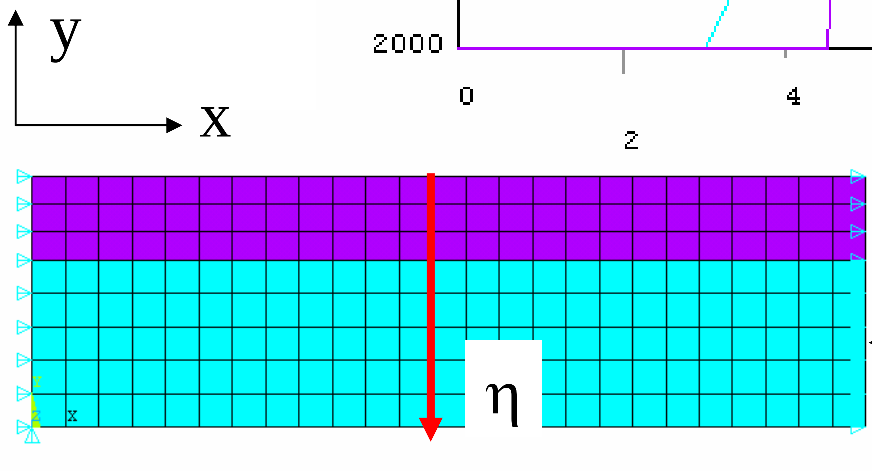
Tensioni σ_y non mediate



Tensioni σ_y mediate

In casi in (
l’operazio
Esempio 1
allungame

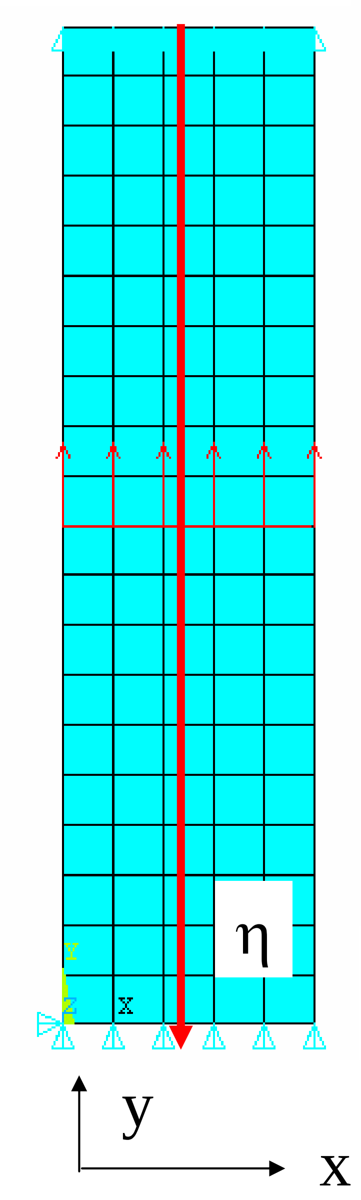




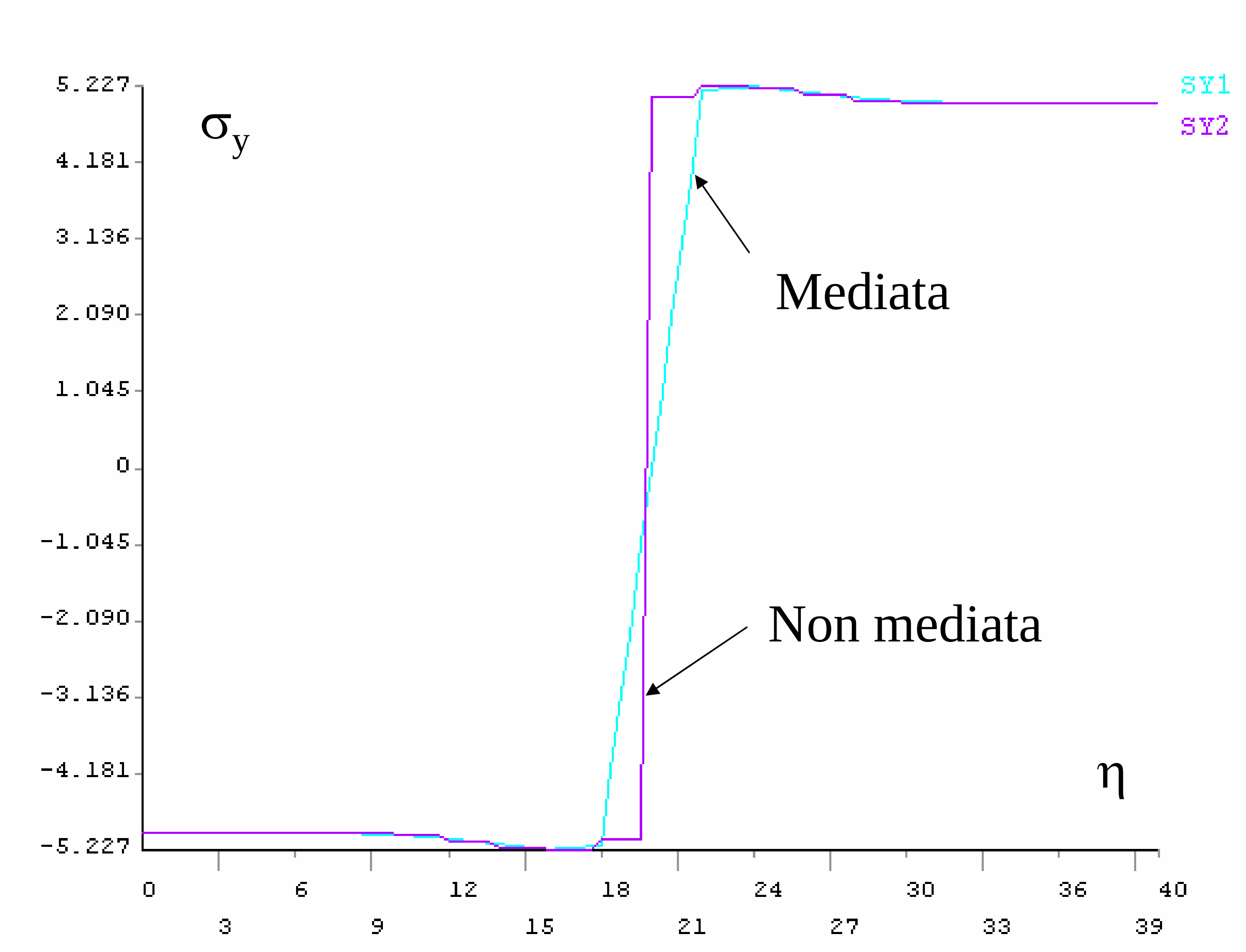
E=10^5 MPa

E=2.1 10^5 MPa

Esempi

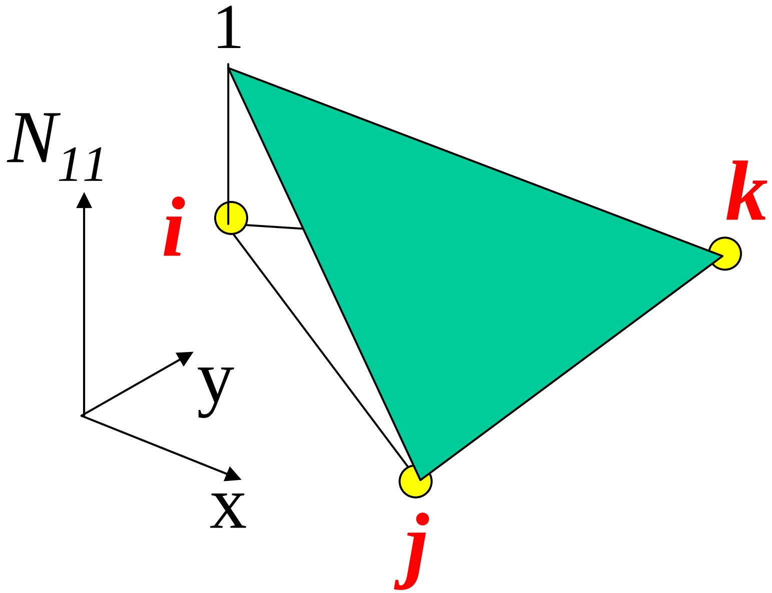


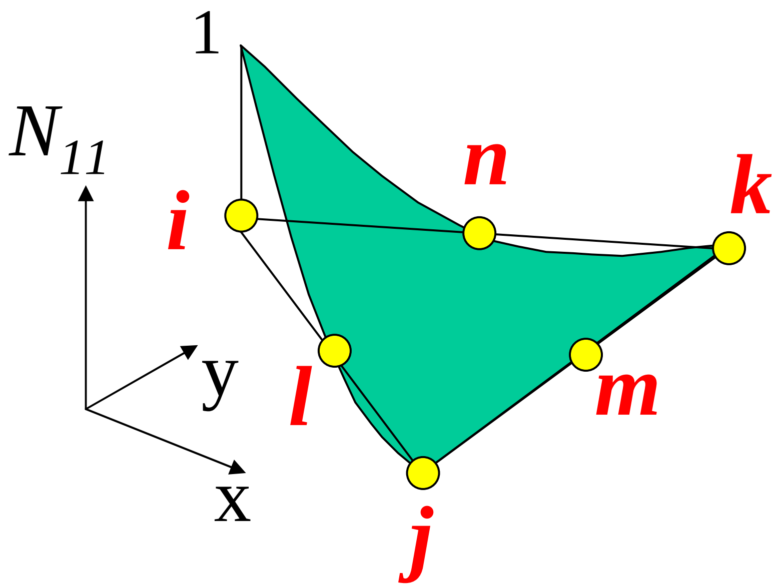
η
y
x



σy
5.227
4.181
3.136
2.090
1.045
0
-1.045
-2.090
-3.136
-4.181
-5.227
SY1
SY2
Mediata
Non mediata
η
0
3
6
9
12
15
18
21
24
27
30
33
36
39
40

Elementi di ordine superiore





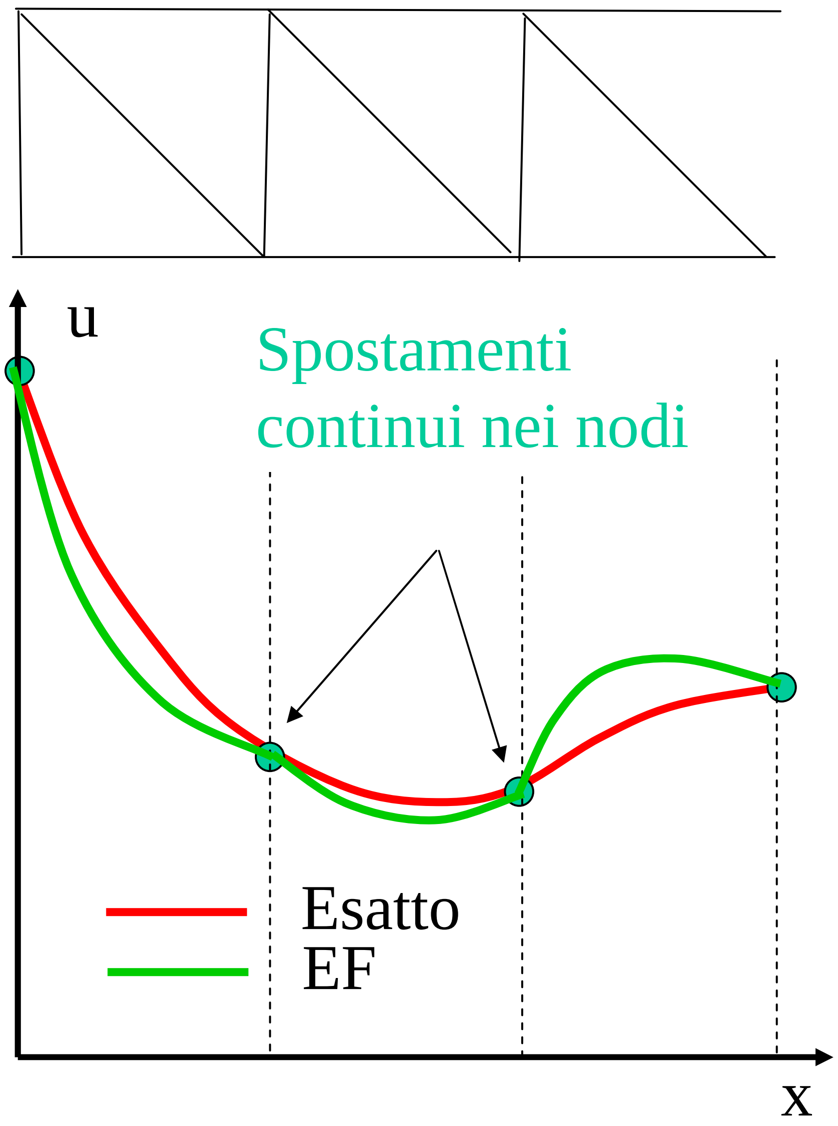
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N^e_{lm}(x,y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

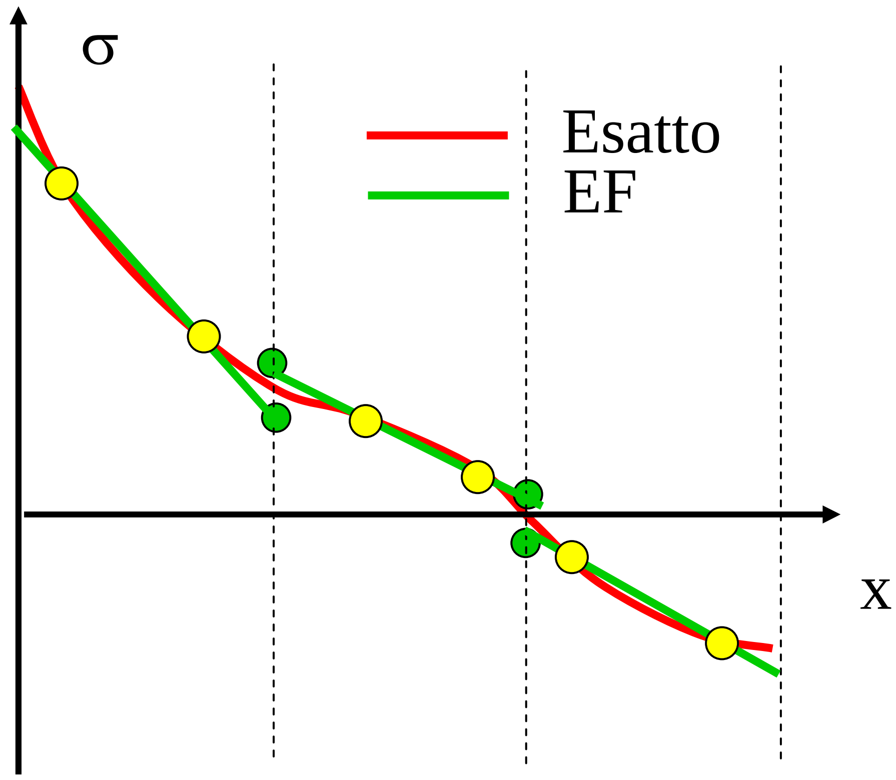
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{11}(x_l, y_l) = 0 \\ N_{11}(x_m, y_m) = 0 \\ N_{11}(x_n, y_n) = 0 \end{cases}$$

$$N^e_{lm}(x,y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y + \\ + D_{lm} \cdot x^2 + E_{lm} \cdot y^2 + F_{lm} \cdot xy$$

Elemento con F.ne Forma quadratica



Tensioni discontinue nei nodi

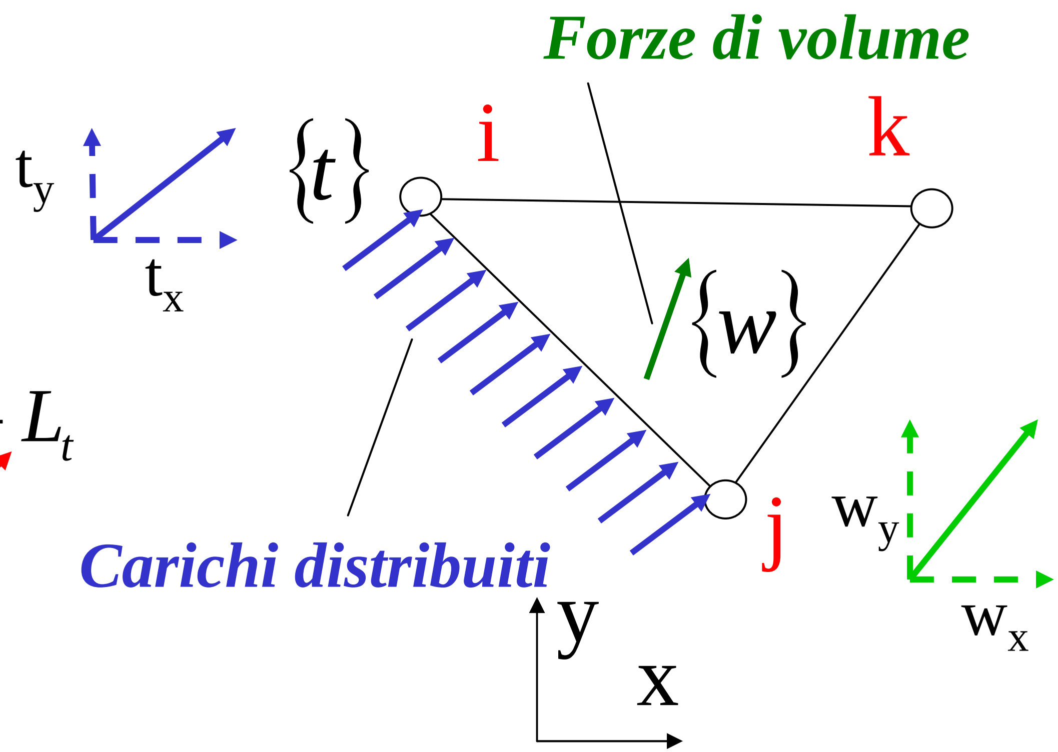


Carichi non concentrati

Lavoro forze di volume

$$L_{est} = \{\delta U^e\}^T \{P^e\} + L_W + L_t$$

Lavoro carichi distribuiti



$$dL_W = \{\delta v\}^T \{w\} dV$$

$$L_W = \int_V \{\delta v\}^T \{w\} dV = \int_V \{\delta U^e\}^T [N]^T \{w\} dV = \{\delta U^e\}^T \int_V [N]^T \{w\} dV$$

$$L_t = \int_L \{\delta v\}^T \{t\} dL = \{\delta U^e\}^T \int_L [N]^T \{t\} dL$$

$$\{P^e\} = [K^e]\{U^e\} + \{P_W^e\} + \{P_t^e\}$$

$$\{P_W^e\} = -\int_V [N]^T \{w\} dV$$

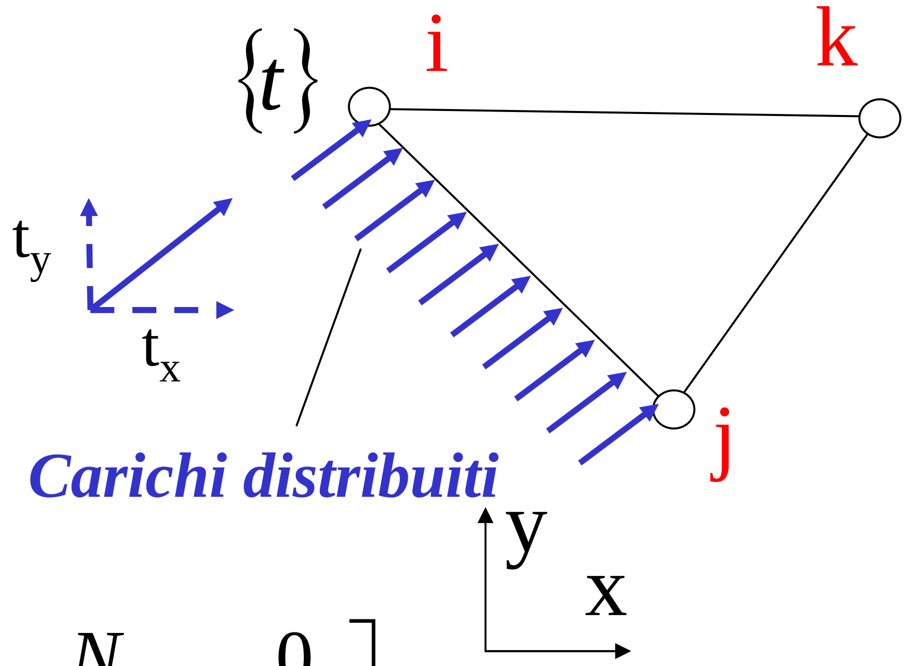
$$\{P_t^e\} = -\int_L [N]^T \{t\} dL$$

Reazioni vincolari conseguenti all'applicazione all'elemento delle forze distribuite e di volume = - carichi che l'elemento trasmette ai nodi in seguito alla presenza delle forze distribuite o di volume (carichi nodali)

Esempio: carico uniformemente distribuito sul lato di un elemento triangolare

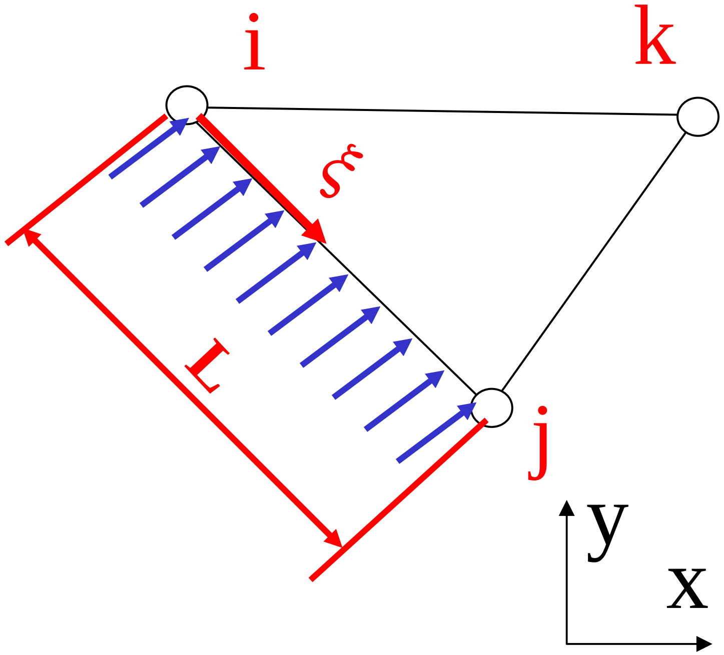
$$\{P_t^e\} = -\int_L [N]^T \{t\} d\xi$$

$6x1 \qquad 6x2 \quad 2x1$



$$[N] = \begin{bmatrix} N_{11} & 0 & N_{13} & 0 & N_{15} & 0 \\ 0 & N_{11} & 0 & N_{13} & 0 & N_{15} \end{bmatrix}$$

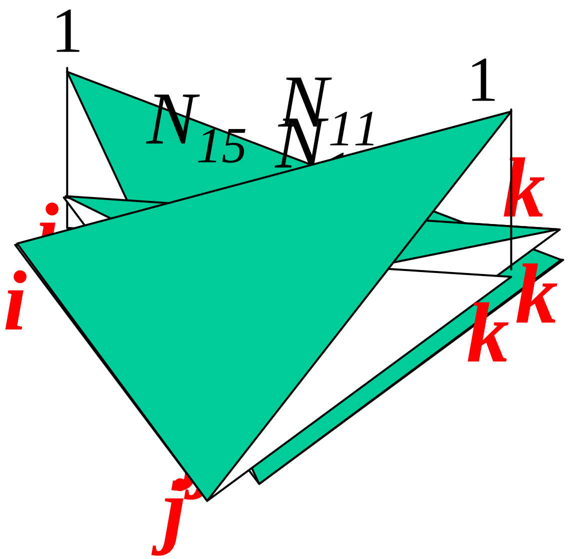
$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



$$p_{t,ix}^e = \int_L N_{11}(x, y \in L)\, t_x \, d\xi = t_x \int_L \frac{L-\xi}{L} d\xi = \frac{t_x L}{2}$$

$$p_{t,jx}^e = \int_L N_{13}(x, y \in L)\, t_x \, dL = t_x \int_L \frac{\xi}{L} dL = \frac{t_x L}{2}$$

$$p_{t,kx}^e = \int_L N_{15}(x, y \in L)\, t_x \, dL = t_x \int_L 0 \, dL = 0$$



Carichi nodali equivalenti