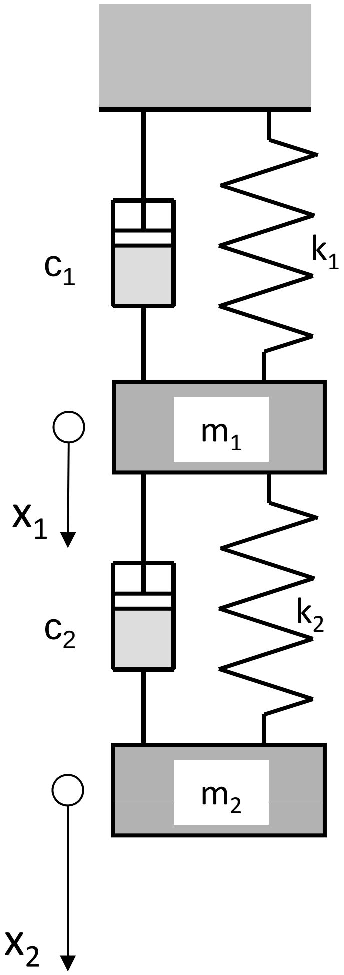
SISTEMA A 2 G.D.L. LIBERO E SMORZATO



Equazioni di equilibrio

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0$$
$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0$$

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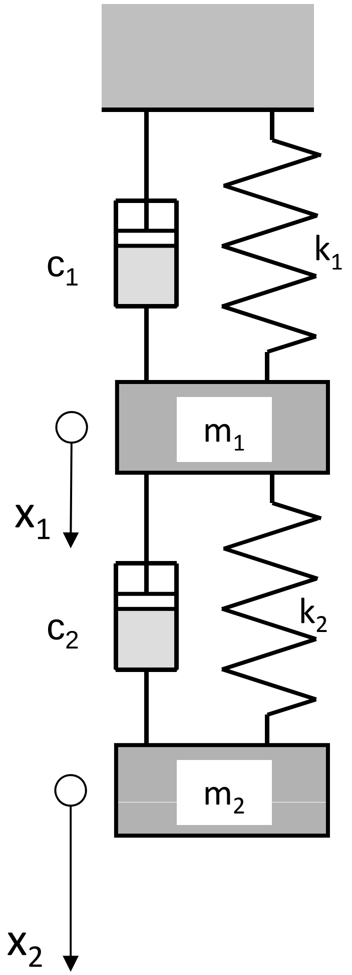


$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=0$$

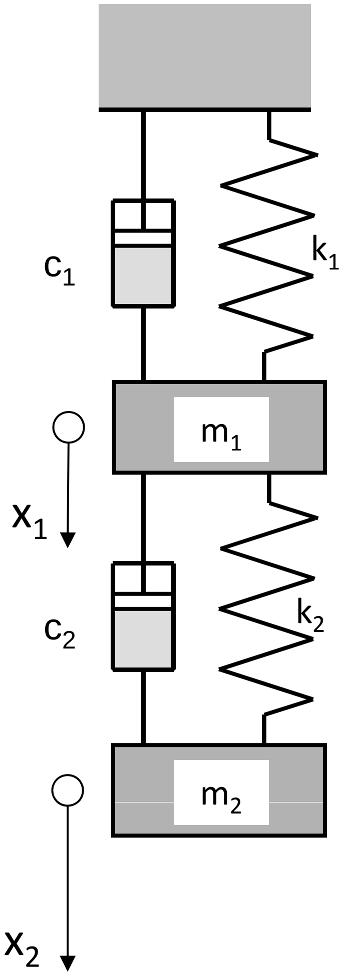
Si assume

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 e^{i\omega t} \\ X_2 e^{i\omega t} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = i\omega \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t}$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t}$$

SISTEMA A 2 G.D.L. LIBERO E SMORZATO



$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=0$$

Sostituendo

$$-\omega^2[M]\{X\}e^{i\omega t}+i\omega[C]\{X\}e^{i\omega t}+[K]\{X\}e^{i\omega t}=0$$

da cui:

$$\left([K]-\omega^2[M]+i\omega[C]\right)\{X\}=0$$

In generale il vettore spostamenti sarà composto da numeri complessi, per cui:

- i vari gdl vibrano con la stessa pulsazione
- i vari gdl possono avere uno sfasamento reciproco

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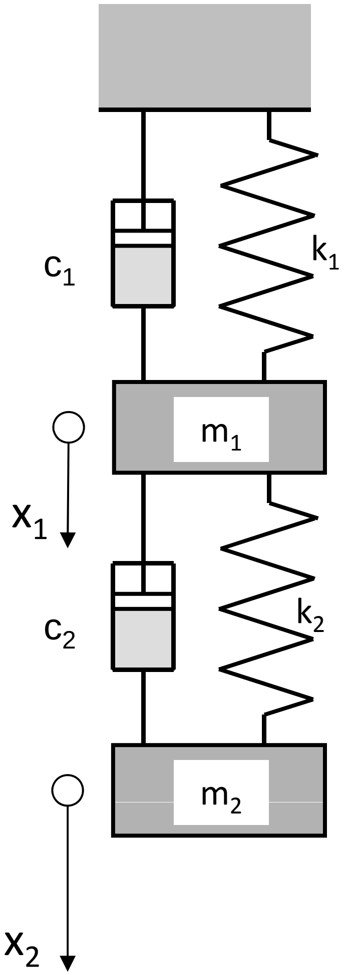


$$\left([K]-\omega^2[M]+i\omega[C]\right)\{X\}=0$$

Sistema lineare omogeneo, per avere sln non banale:

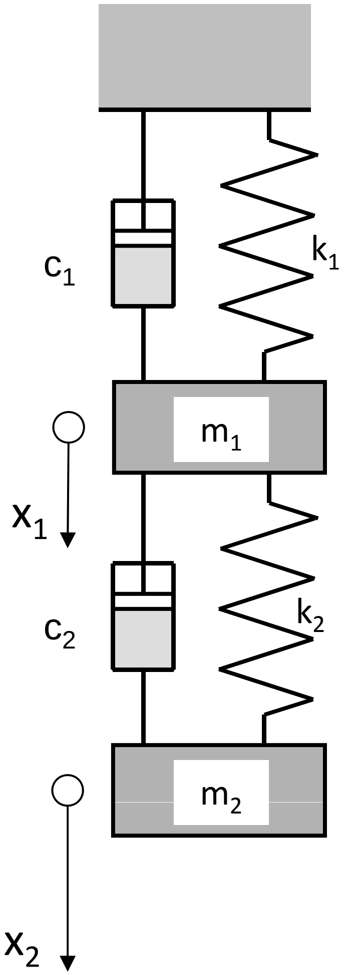
$$\det\begin{bmatrix} k_1+k_2+i\omega(c_1+c_2)-\omega^2 m_1 & -k_2-i\omega c_2 \\ -k_2-i\omega c_2 & k_2+i\omega c_2-\omega^2 m_2 \end{bmatrix}=0$$

$$\left(k_1+k_2+i\omega(c_1+c_2)-\omega^2 m_1\right)\left(k_2+i\omega c_2-\omega^2 m_2\right)-\left(k_2+i\omega c_2\right)^2=0$$

In generale, radici in campo complesso:

$$\omega_j=\omega_s^{(j)}+i\omega_n^{(j)}$$

SISTEMA A 2 G.D.L. LIBERO E SMORZATO



$$\omega_j = \omega_s^{(j)} + i\omega_n^{(j)}$$

Soluzione del tipo:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1^{(J)} \\ X_2^{(j)} \end{Bmatrix} e^{-\omega_n^{(j)}t} e^{-i\omega_s^{(j)}t}$$

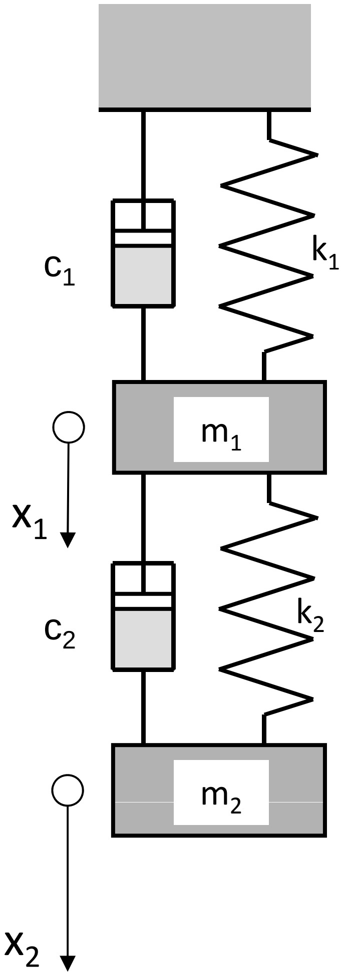
Andamento armonico smorzato

Forme modali

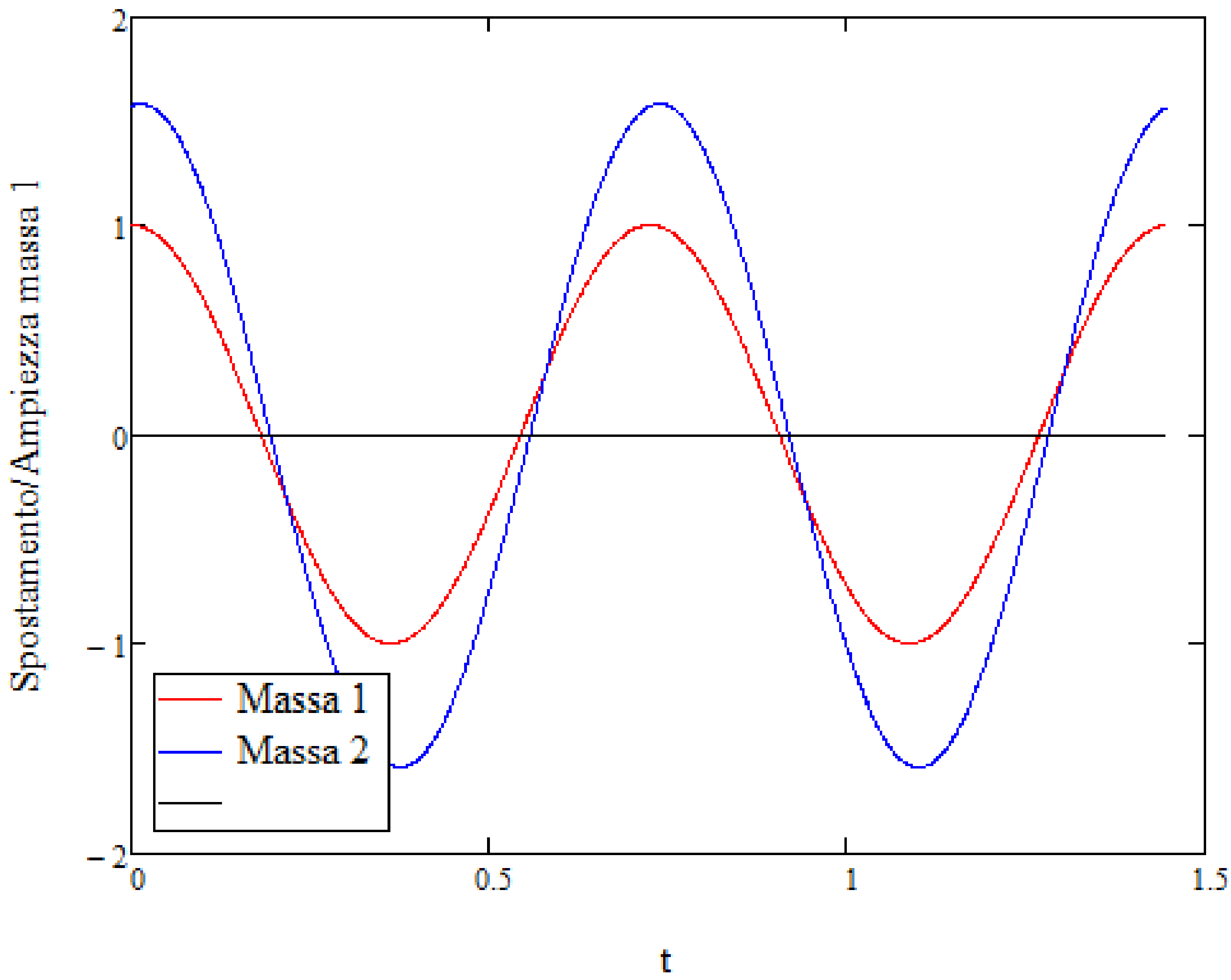
$$r_j = \frac{X_2^{(j)}}{X_1^{(j)}} = \frac{k_1 + k_2 + i\omega_j(c_1 + c_2) - \omega^2 m_1}{k_2 + i\omega_j c_2}$$

Rapporto con ampiezza e fase

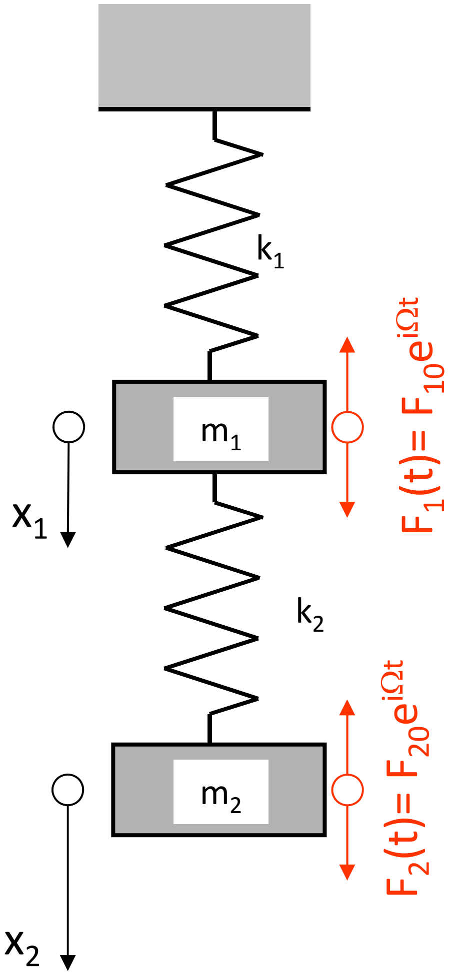
SISTEMA A 2 G.D.L. LIBERO E SMORZATO



Esempio di oscillazione libera secondo uno dei modi propri



SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



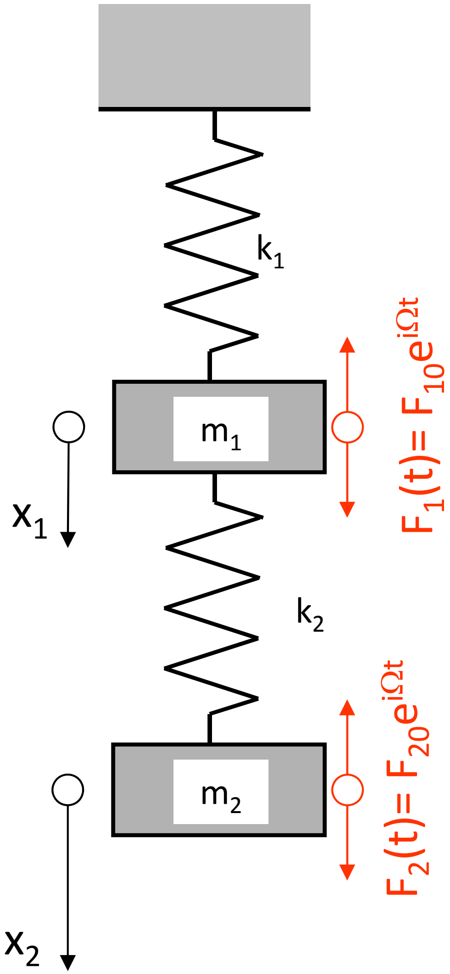
Equazioni di equilibrio

$$m_1\ddot{x}_1 + k_1 x_1 + k_2\left(x_1 - x_2\right) = F_{10}e^{1\Omega t}$$
$$m_2\ddot{x}_2 + k_2\left(x_2 - x_1\right) = F_{20}e^{1\Omega t}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}e^{i\Omega t}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\}e^{i\Omega t}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



$$[M]\{\ddot{x}\}+[K]\{x\}=\{F\}e^{i\Omega t}$$

Soluzione “a regime”, dopo esaurimento del transitorio iniziale

⇓

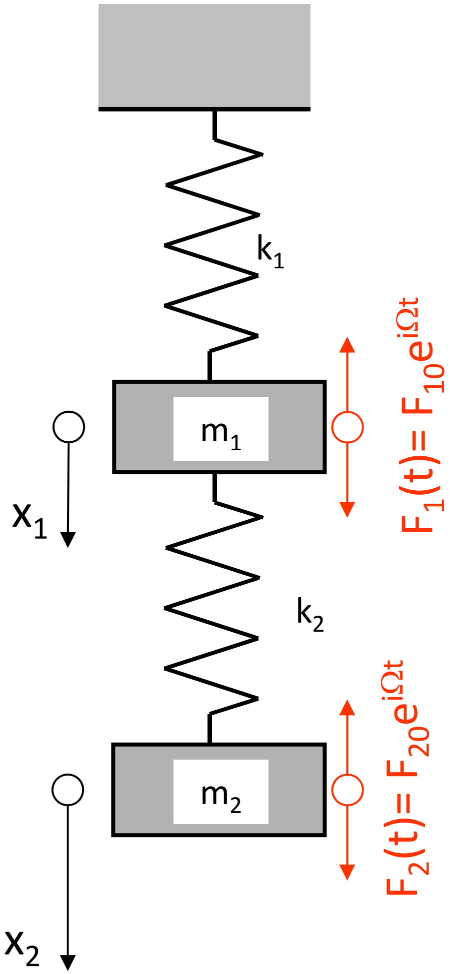
Soluzione = Integrale particolare sistema non omogeneo

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 e^{i\Omega t} \\ X_2 e^{i\Omega t} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = \{X\} e^{i\Omega t}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = i\Omega \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = i\Omega \{X\} e^{i\Omega t}$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\Omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = -\Omega^2 \{X\} e^{i\Omega t}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



Sostituendo

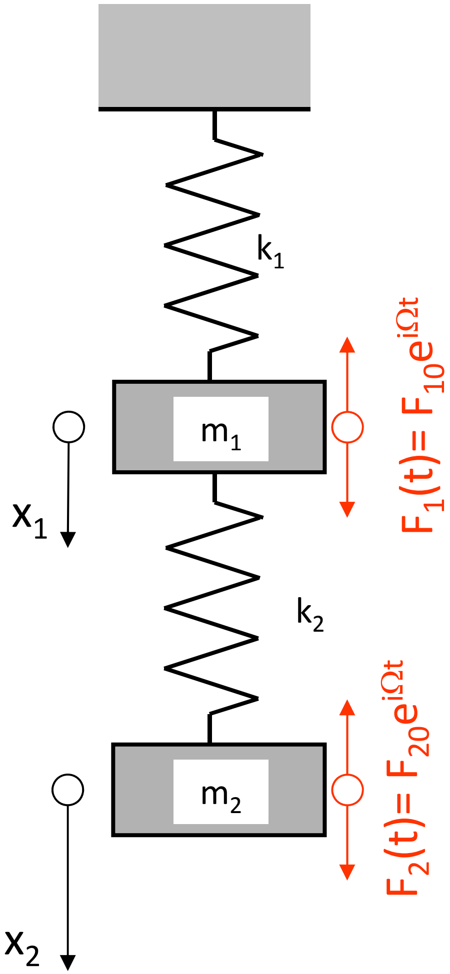
$$-\Omega^2[M]\{X\}e^{i\Omega t}+[K]\{X\}e^{i\Omega t}=\{F\}e^{i\Omega t}$$

da cui

$$\left([K]-\Omega^2[M]\right)\{X\}=\{F\}$$

$$\begin{bmatrix} k_1+k_2-\Omega^2 m_1 & -k_2 \\ -k_2 & k_2-\Omega^2 m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



$$\begin{bmatrix} k_1 + k_2 - \Omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \Omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

posto

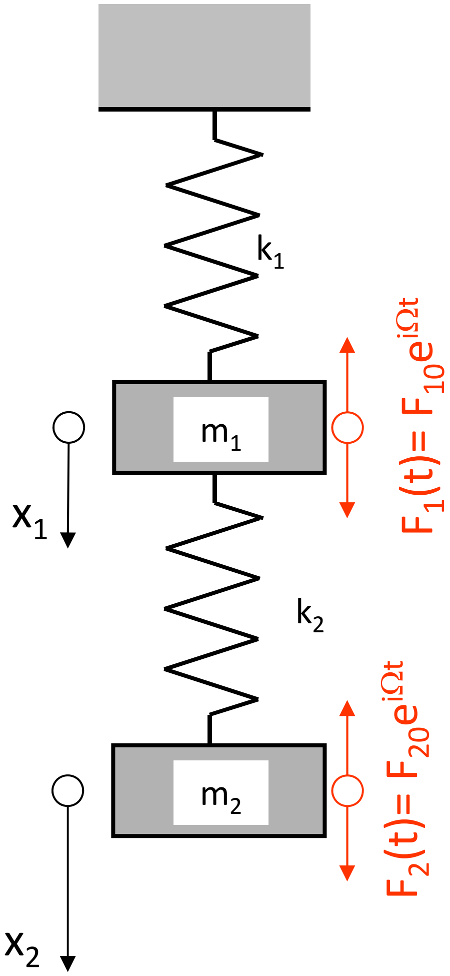
$$\Delta = \left(k_1 + k_2 - \Omega^2 m_1\right)\left(k_2 - \Omega^2 m_2\right) - \left(k_2\right)^2$$

si ottiene:

$$X_1(\Omega_0) := \frac{\left(k_2 - \Omega_0{}^2 \cdot m_2\right) \cdot F_{10} + k_2 \cdot F_{20}}{\left(k_1 + k_2 - \Omega_0{}^2 \cdot m_1\right) \cdot \left(k_2 - \Omega_0{}^2 \cdot m_2\right) - k_2{}^2}$$

$$X_2(\Omega_0) := \frac{k_2 \cdot F_{10} + \left(k_1 + k_2 - \Omega_0{}^2 \cdot m_1\right) \cdot F_{20}}{\left(k_1 + k_2 - \Omega_0{}^2 \cdot m_1\right) \cdot \left(k_2 - \Omega_0{}^2 \cdot m_2\right) - k_2{}^2}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



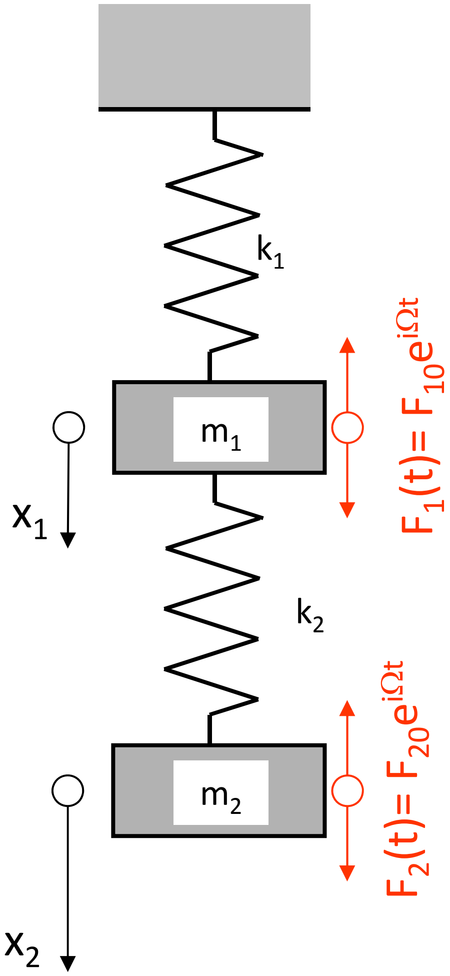
definendo: $\omega_1 := \sqrt{\dfrac{k_1}{m_1}}$ $\omega_2 := \sqrt{\dfrac{k_2}{m_2}}$

si possono porre nella forma:

$$X_1(\Omega_0) := \frac{\left(1 - \dfrac{\Omega_0^{\,2}}{\omega_2^{\,2}}\right)\cdot\dfrac{F_{10}}{k_1} + \dfrac{F_{20}}{k_1}}{\left[\left(\dfrac{k_2}{k_1} + 1 - \dfrac{\Omega_0^{\,2}}{\omega_1^{\,2}}\right)\cdot\left(1 - \dfrac{\Omega_0^{\,2}}{\omega_2^{\,2}}\right) - \dfrac{k_2}{k_1}\right]}$$

$$X_2(\Omega_0) := \frac{\dfrac{F_{10}}{k_1} + \left(1 + \dfrac{k_2}{k_1} - \dfrac{\Omega_0^{\,2}}{\omega_1^{\,2}}\right)\cdot\dfrac{F_{20}}{k_2}}{\left[\left(\dfrac{k_2}{k_1} + 1 - \dfrac{\Omega_0^{\,2}}{\omega_1^{\,2}}\right)\cdot\left(1 - \dfrac{\Omega_0^{\,2}}{\omega_2^{\,2}}\right) - \dfrac{k_2}{k_1}\right]}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



L'andamento può essere studiato per un caso particolare, quale: $m_1=m_2= m$; $k_1=k_2= k$; $F_{10}=F_{20}= F_0$, ponendo

$$\omega_0 = \sqrt{\frac{k}{m}}$$

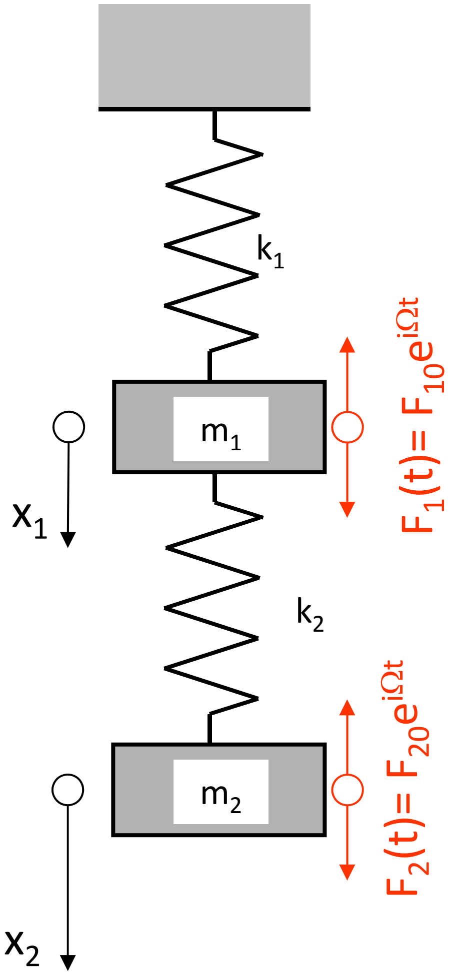
$$r_f = \frac{\Omega}{\omega_0}$$

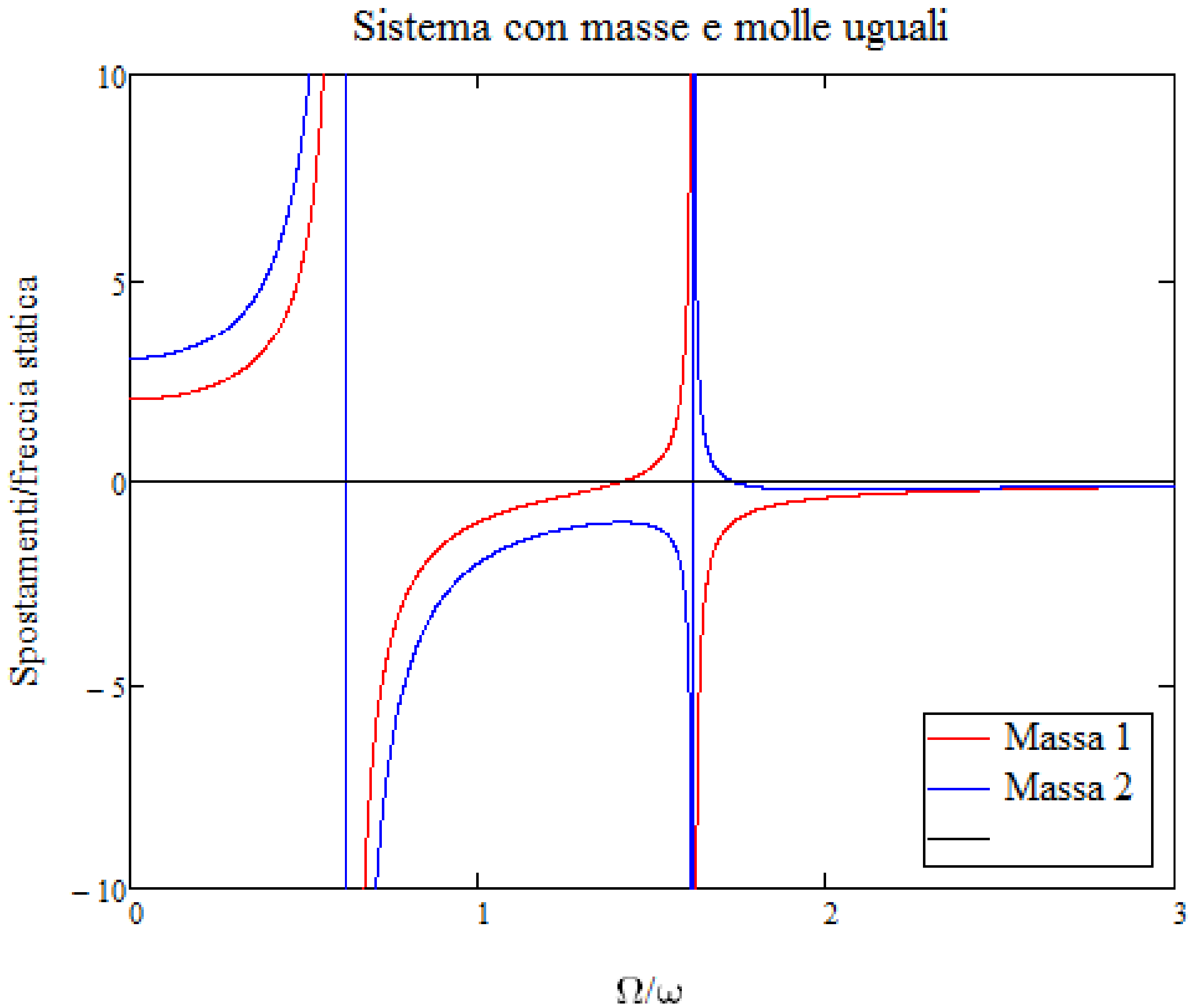
E normalizzando gli spostamenti rispetto alla freccia statica:

$$\frac{X_1}{F_0/k} = \frac{\left(1-r_f^2\right)+1}{\left[\left(2-r_f^2\right)\left(1-r_f^2\right)-1\right]}$$

$$\frac{X_2}{F_0/k} = \frac{\left(2-r_f^2\right)+1}{\left[\left(2-r_f^2\right)\left(1-r_f^2\right)-1\right]}$$

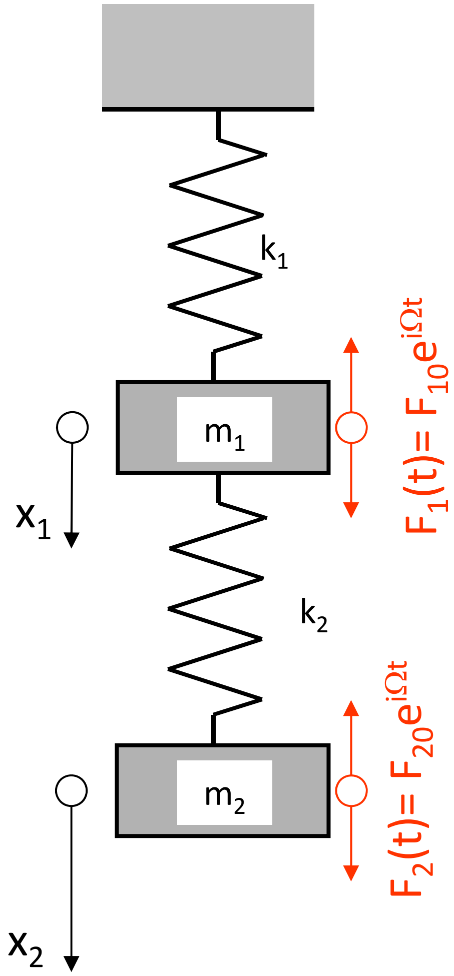
SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA

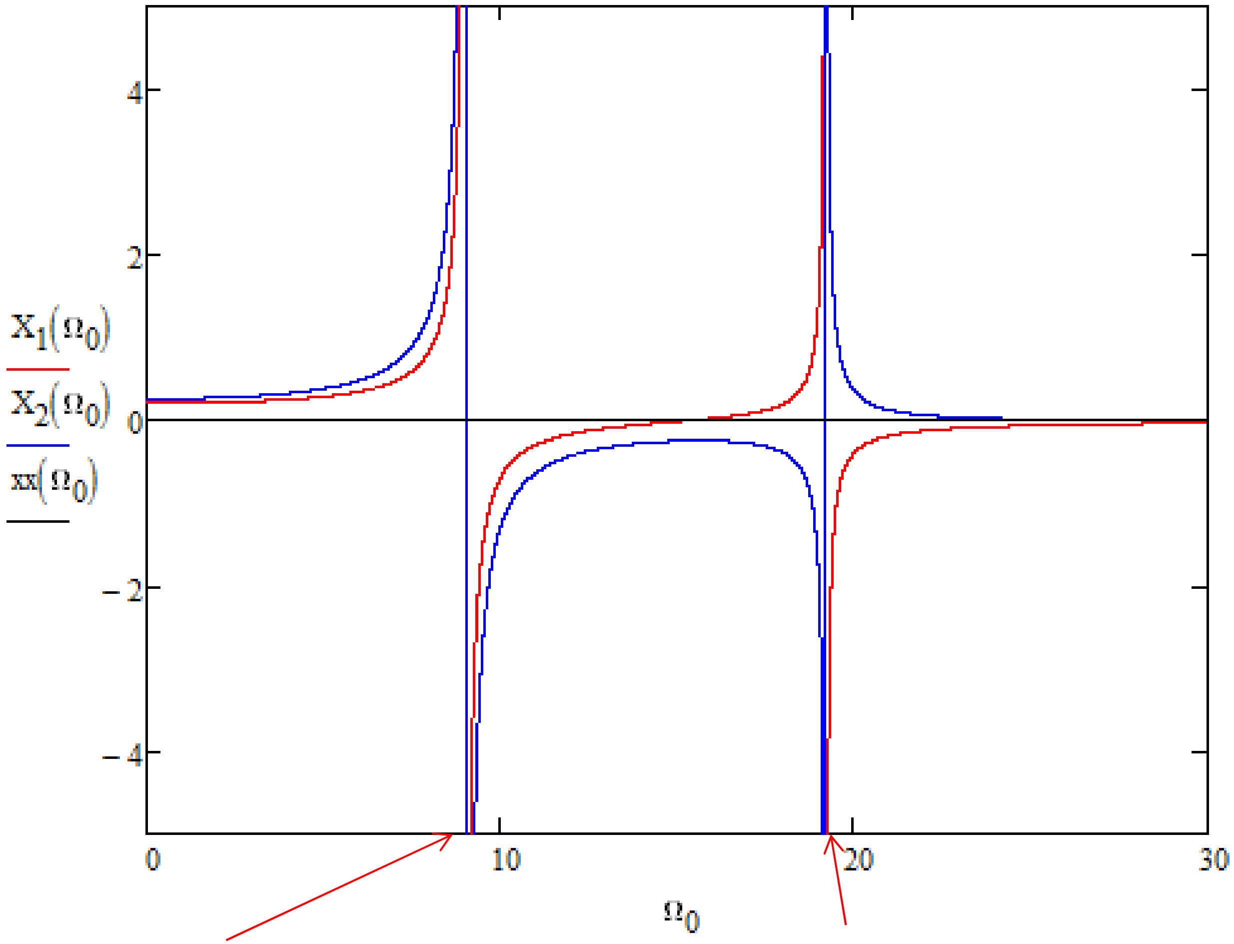




SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA

Andamenti simili si ottengono in un caso generale, quale:
m_1=10 kg, m_2= 5 kg, k_1=1500 N/m, k_2= 1000 N/m
F_{10}=250 N, F_{20}=50 N

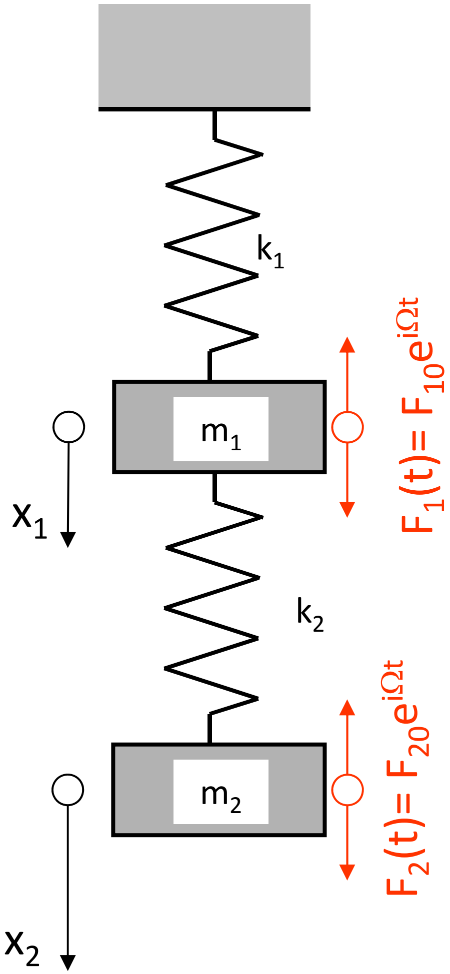




$$\omega_{1n} = 9.021\ rad/s \qquad \omega_{2n} = 19.199\ rad/s$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA

Freccia statica prodotta dalle forze

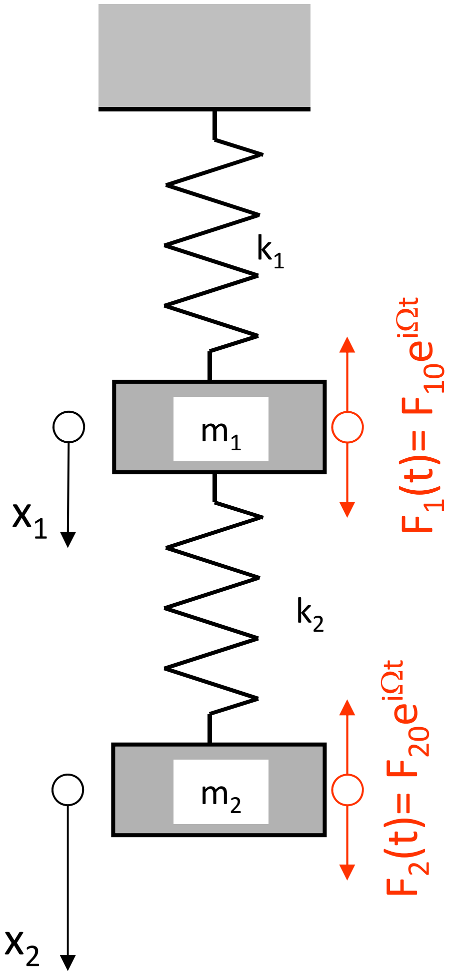


$$\delta_1 := \frac{F_{10} + F_{20}}{k_1} \qquad \delta_2 := \frac{F_{10} + F_{20}}{k_1} + \frac{F_{20}}{k_2}$$

$$\frac{\delta_2}{\delta_1} := \frac{\dfrac{F_{10} + F_{20}}{k_1} + \dfrac{F_{20}}{k_2}}{\dfrac{F_{10}+F_{20}}{k_1}} = 1 + \frac{\dfrac{F_{20}}{k_2}}{\dfrac{F_{10}+F_{20}}{k_1}} =$$

$$1 + \frac{k_1}{k_2} \cdot \frac{F_{20}}{F_{10} + F_{20}} = 1 + \frac{k_1}{k_2} \cdot \frac{1}{\dfrac{F_{10}}{F_{20}} + 1}$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA



Uguagliando al rapporto tra gli spostamenti nelle forme modali:

$$\frac{\delta_2}{\delta_1} := r_j$$

si ottiene:

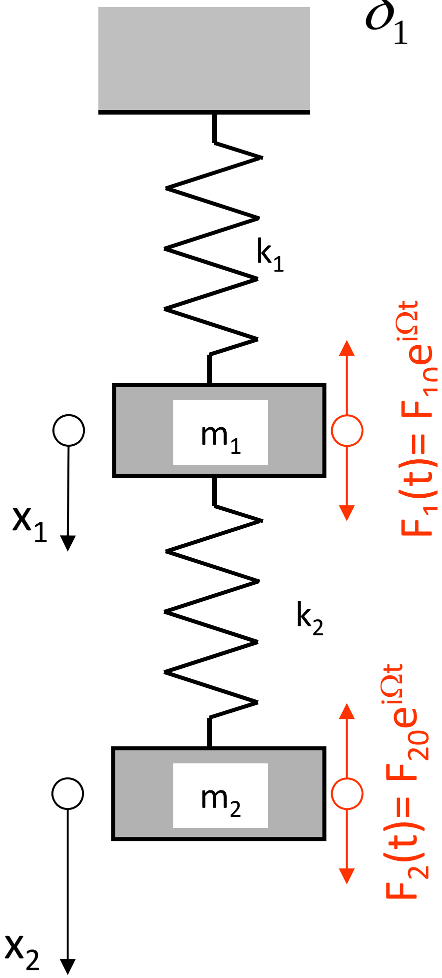
$$r_j - 1 := \frac{k_1}{k_2} \cdot \frac{1}{\frac{F_{10}}{F_{20}} + 1}$$

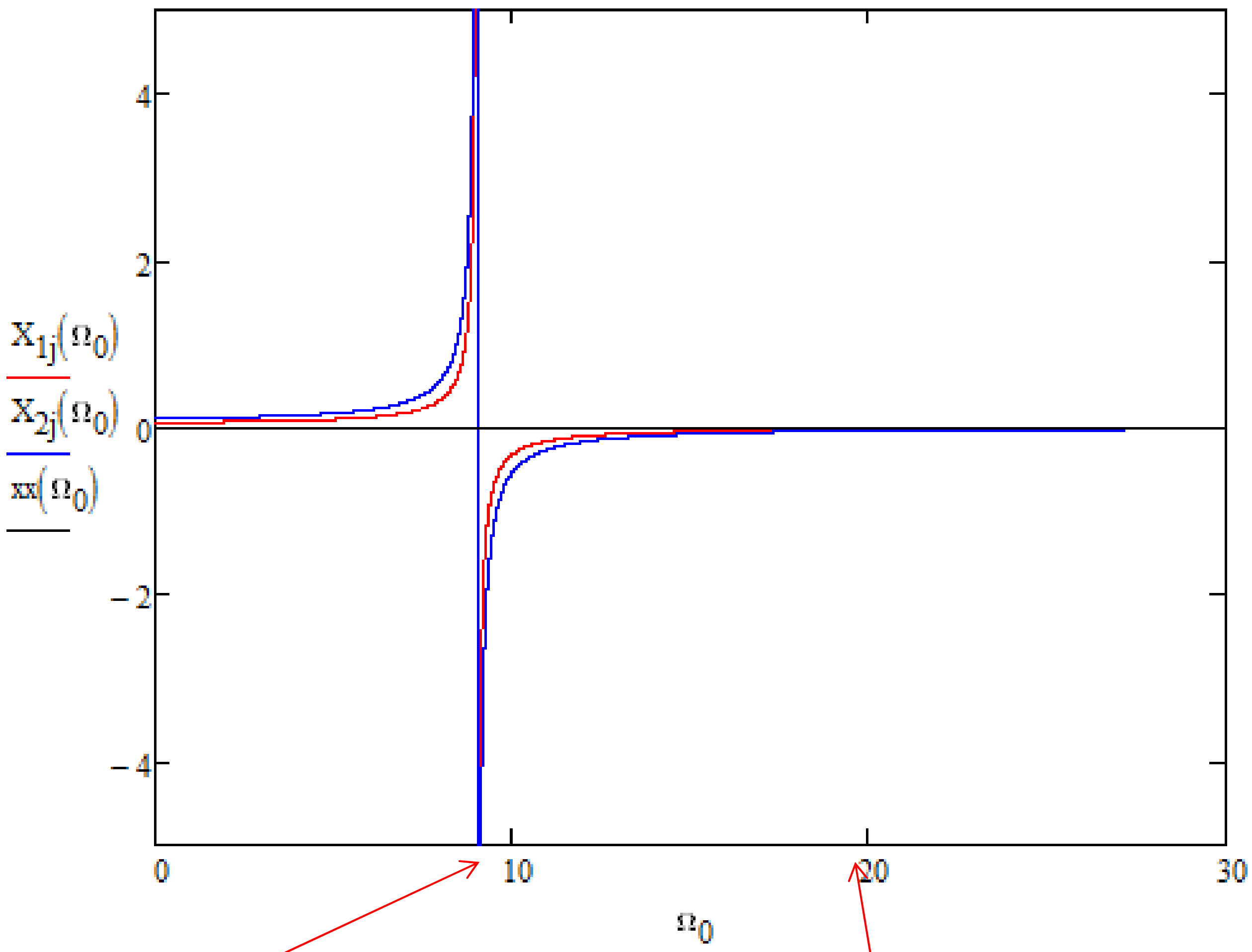
da cui:

$$\left(\frac{F_{10}}{F_{20}}\right)^{(j)} := \frac{k_1}{k_2} \cdot \frac{1}{r_j - 1} - 1$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA

$$\frac{\delta_2}{\delta_1} = r_1 \qquad \{1 \quad r_1\}\begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} \neq 0 \qquad \{1 \quad r_2\}\begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = 0$$

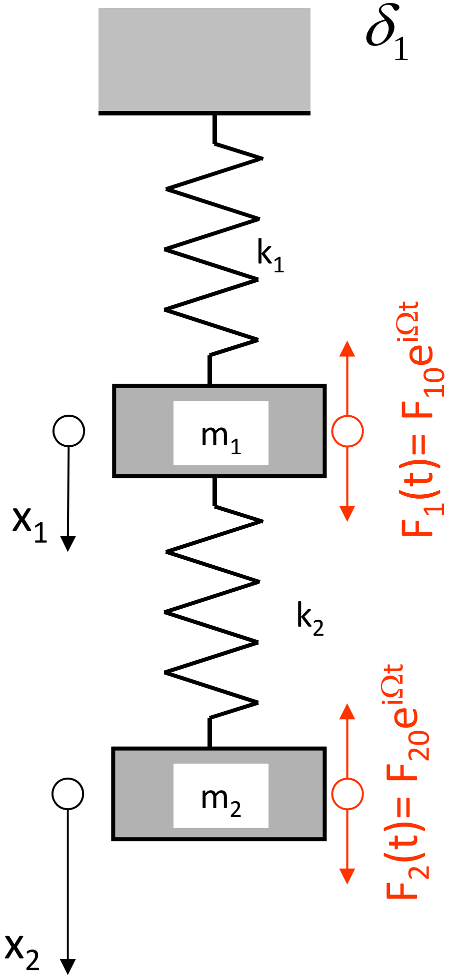


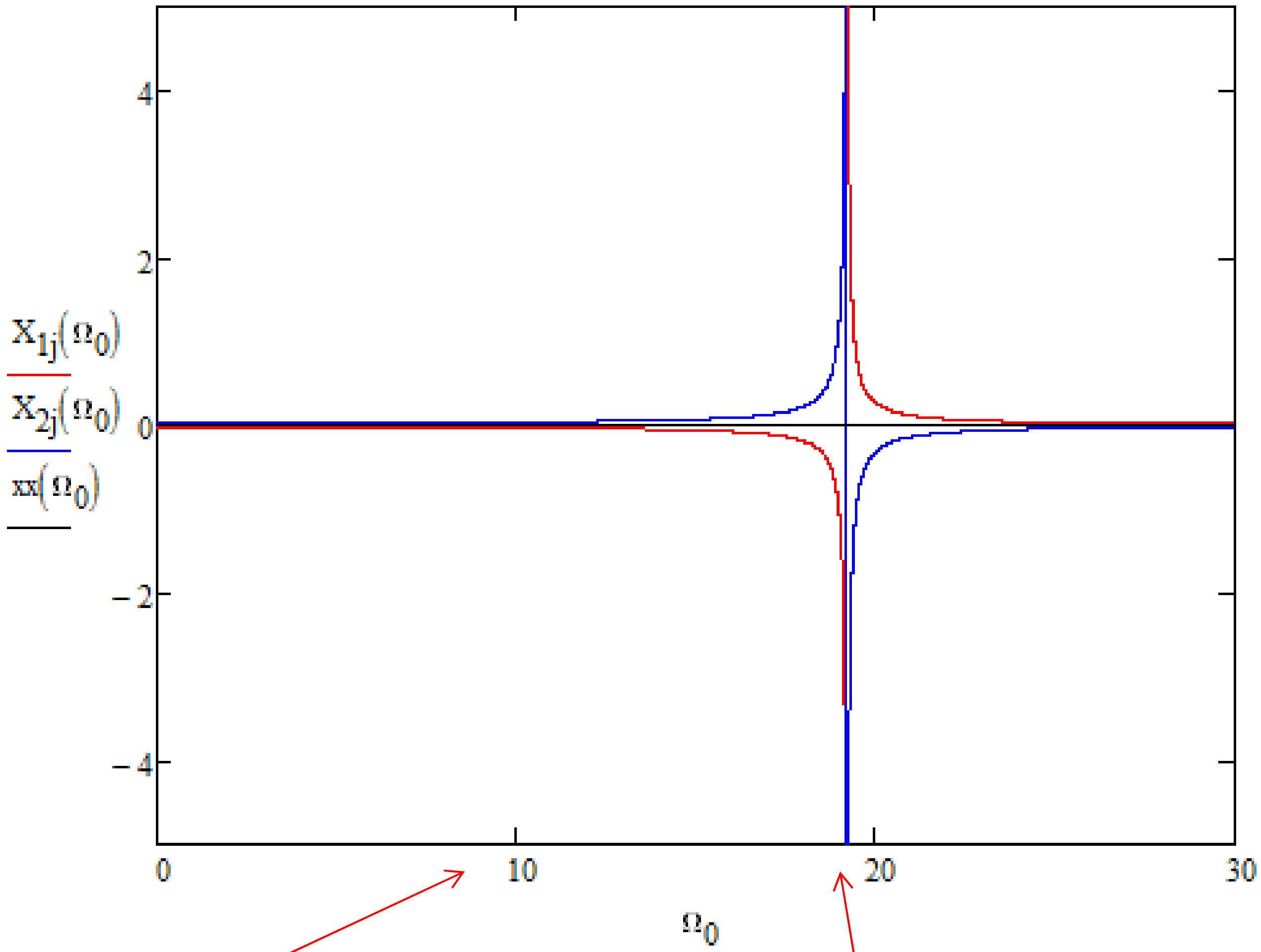


$$\omega_{1n} = 9.021\ rad/s \qquad \omega_{2n} = 19.199\ rad/s$$

SISTEMA A 2 G.D.L. NON SMORZATO CON FORZANTE ESTERNA

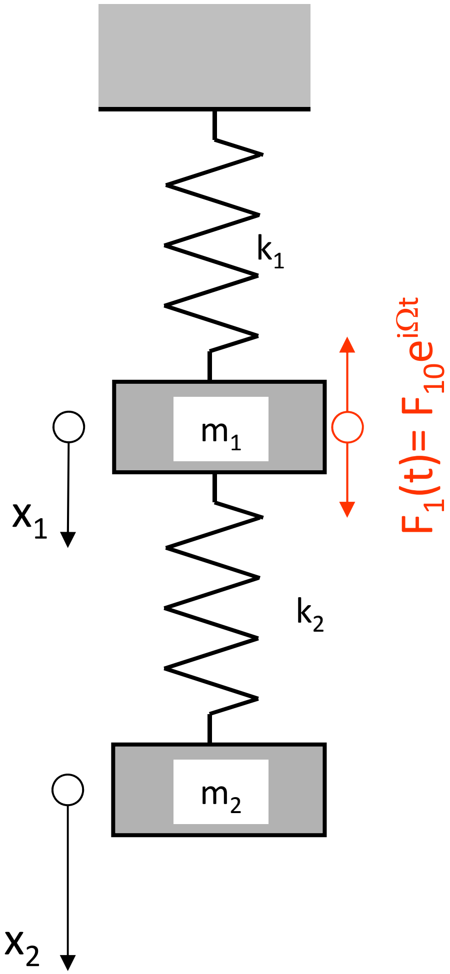
$$\frac{\delta_2}{\delta_1} = r_2 \qquad \begin{Bmatrix} 1 & r_1 \end{Bmatrix} \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = 0 \qquad \begin{Bmatrix} 1 & r_2 \end{Bmatrix} \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} \neq 0$$





$$\omega_{1n} = 9.021 \ rad/s \qquad \omega_{2n} = 19.199 \ rad/s$$

SISTEMA A 2 G.D.L. NON SM.CON FORZANTE - SMORZATORE DINAMICO



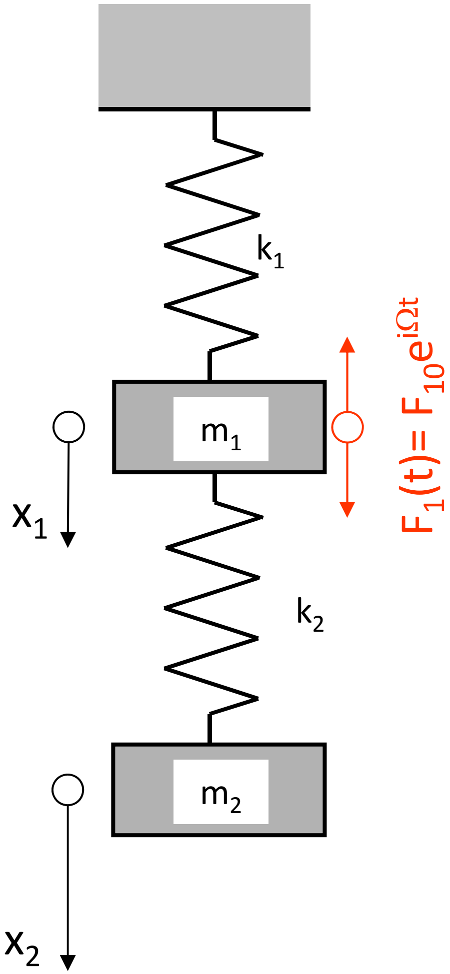
Si vuole studiare il caso:

$$\sqrt{\frac{k_1}{m_1}} := \sqrt{\frac{k_2}{m_2}} = \omega_0 \qquad F_{20} := 0$$

$$X_{1sd}(\Omega_0) := \frac{\left(1 - \frac{\Omega_0^{\,2}}{\omega_0^{\,2}}\right)\cdot\frac{F_{10}}{k_1}}{\left[\left(\frac{k_2}{k_1} + 1 - \frac{\Omega_0^{\,2}}{\omega_0^{\,2}}\right)\cdot\left(1 - \frac{\Omega_0^{\,2}}{\omega_0^{\,2}}\right) - \frac{k_2}{k_1}\right]}$$

$$X_{2sd}(\Omega_0) := \frac{\frac{F_{10}}{k_1}}{\left[\left(\frac{k_2}{k_1} + 1 - \frac{\Omega_0^{\,2}}{\omega_0^{\,2}}\right)\cdot\left(1 - \frac{\Omega_0^{\,2}}{\omega_0^{\,2}}\right) - \frac{k_2}{k_1}\right]}$$

SISTEMA A 2 G.D.L. NON SM.CON FORZANTE - SMORZATORE DINAMICO



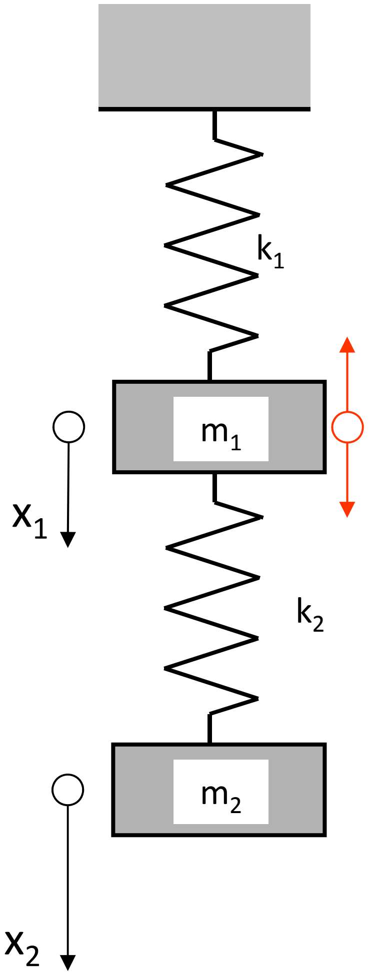
Ponendo: $r_f = \frac{\Omega}{\omega_0}$

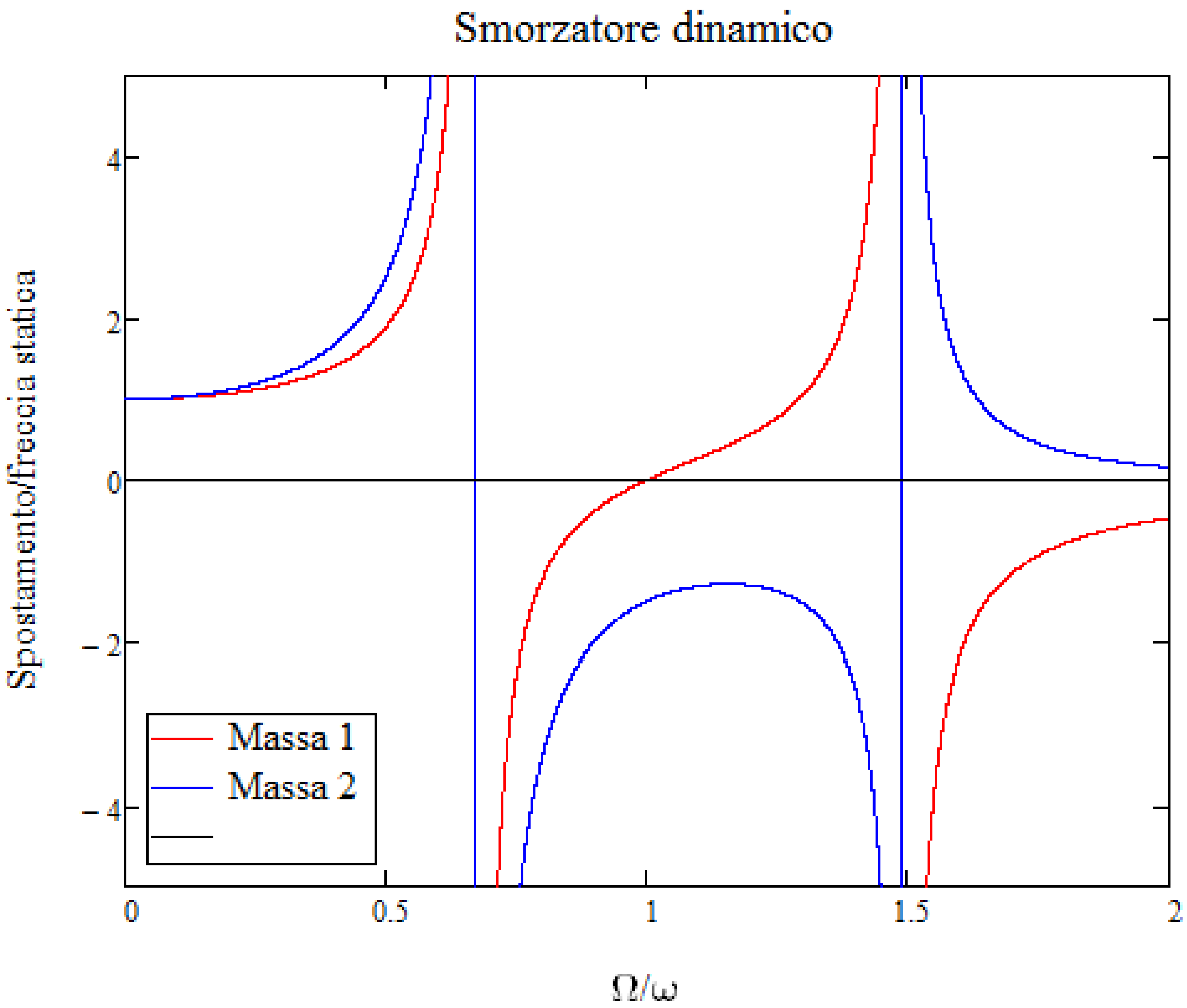
e normalizzando rispetto alla freccia statica: $\frac{F_{10}}{k_1}$

$$\frac{X_1}{F_{10}/k_1} = \frac{\left(1 - r_f^2\right)}{\left(\frac{k_2}{k_1} + 1 - r_f^2\right)\left(1 - r_f^2\right) - \frac{k_2}{k_1}}$$

$$\frac{X_2}{F_{10}/k_1} = \frac{1}{\left(\frac{k_2}{k_1} + 1 - r_f^2\right)\left(1 - r_f^2\right) - \frac{k_2}{k_1}}$$

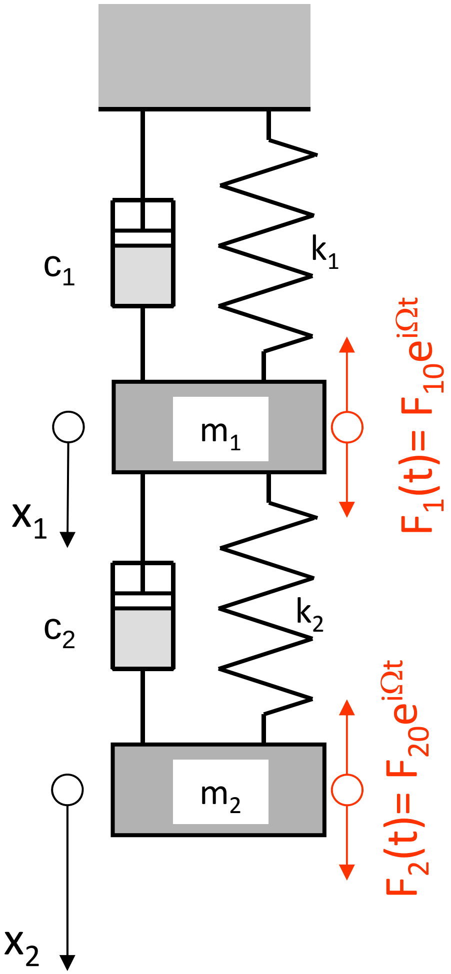
SISTEMA A 2 G.D.L. NON SM.CON FORZANTE - SMORZATORE DINAMICO





In risonanza lo spostamento della massa 1 è nullo.

SISTEMA A 2 G.D.L. SMORZATO CON FORZANTE ESTERNA



Forzanti aventi identica pulsazione, con possibile differenza tra loro di ampiezza e fase

Equazioni di equilibrio

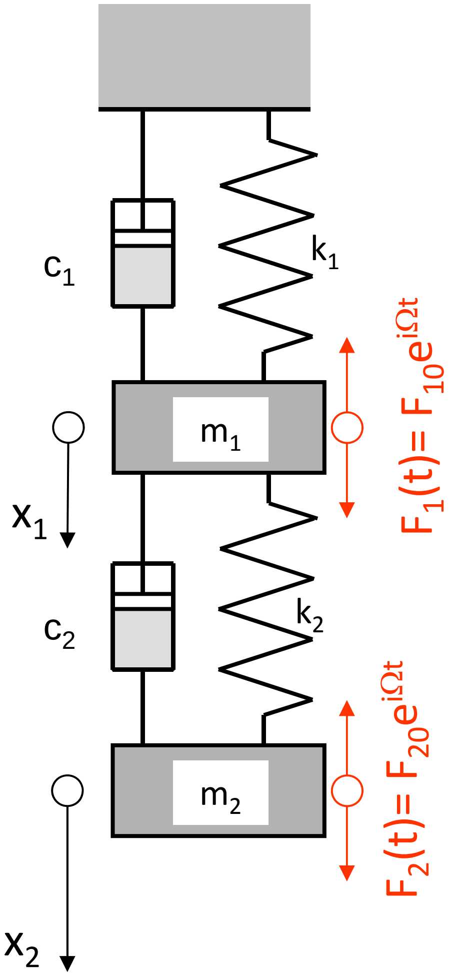
$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2(x_1 - x_2) = F_{10}e^{1\Omega t}$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = F_{20}e^{1\Omega t}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} e^{i\Omega t}$$

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\}e^{i\Omega t}$$

SISTEMA A 2 G.D.L. SMORZATO CON FORZANTE ESTERNA



$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}e^{i\Omega t}$$

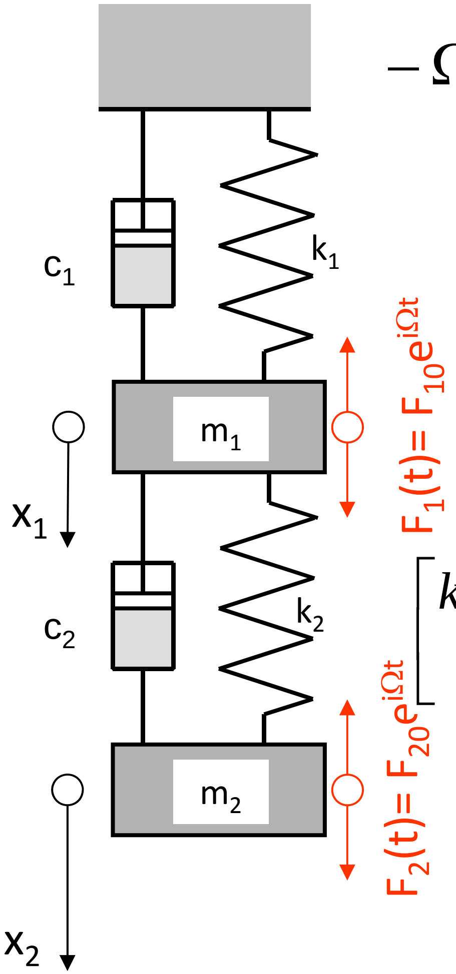
Soluzione a regime = integrale particolare sistema non omogeneo.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 e^{i\Omega t} \\ X_2 e^{i\Omega t} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = \{X\} e^{i\Omega t}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = i\Omega \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = i\Omega \{X\} e^{i\Omega t}$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\Omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\Omega t} = -\Omega^2 \{X\} e^{i\Omega t}$$

SISTEMA A 2 G.D.L. SMORZATO CON FORZANTE ESTERNA



Sostituendo:

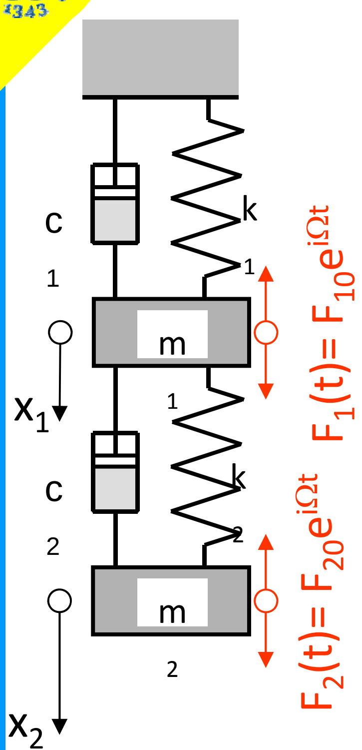
$$-\Omega^2[M]\{X\}e^{i\Omega t} + i\Omega[C]\{X\}e^{i\Omega t} + [K]\{X\}e^{i\Omega t} = \{F\}e^{i\Omega t}$$

$$([K]-\Omega^2[M]+i\Omega[C])\{X\}=\{F\}$$

Sistema lineare non omogeneo

$$\begin{bmatrix} k_1+k_2+i\Omega(c_1+c_2)-\Omega^2 m_1 & -k_2-i\Omega c_2 \\ -k_2-i\Omega c_2 & k_2+i\Omega c_2-\Omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

SISTEMA A 2 G.D.L. SMORZATO CON FORZANTE ESTERNA



$$\begin{bmatrix} k_1+k_2+i\Omega(c_1+c_2)-\Omega^2 m_1 & -k_2-i\Omega c_2 \\ -k_2-i\Omega c_2 & k_2+i\Omega c_2-\Omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

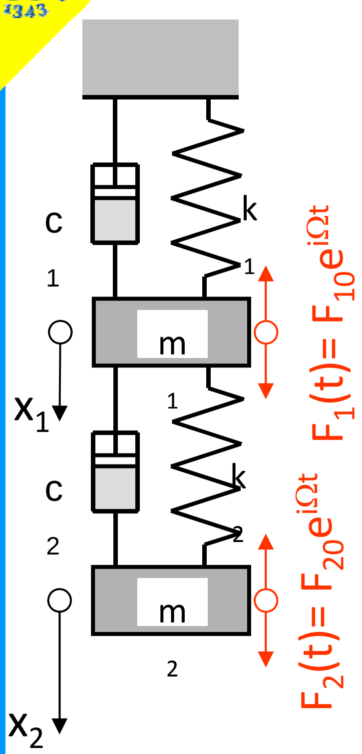
Posto:

$$\Delta = \left(k_1+k_2+i\Omega(c_1+c_2)-\Omega^2 m_1\right)\left(k_2+i\Omega c_2-\Omega^2 m_2\right)-\left(k_2+i\Omega c_2\right)^2$$

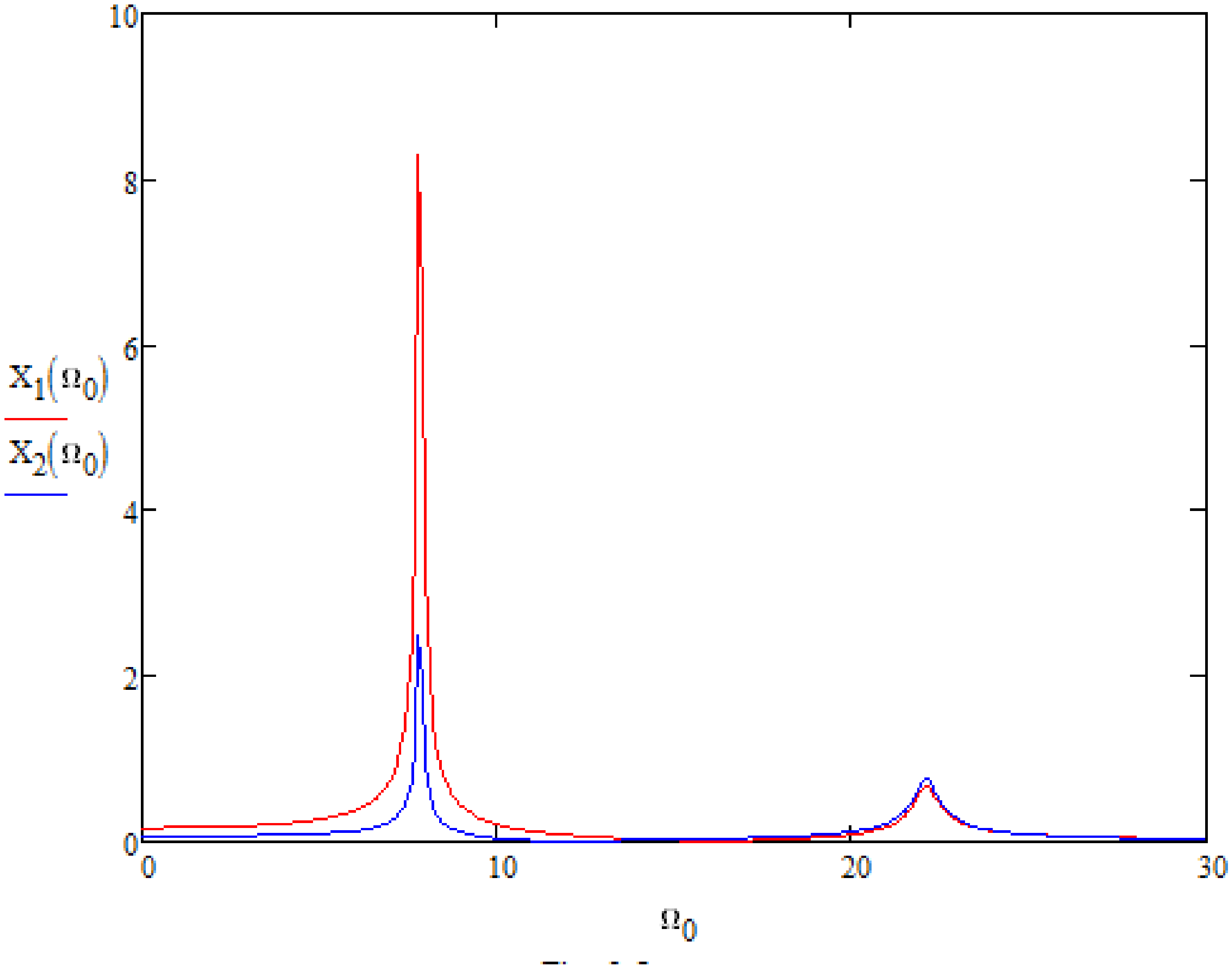
si ottiene:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{\Delta}\begin{bmatrix} k_2+i\Omega c_2-\Omega^2 m_2 & k_2+i\Omega c_2 \\ k_2+i\Omega c_2 & k_1+k_2+i\Omega(c_1+c_2)-\Omega^2 m_1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

SISTEMA A 2 G.D.L. SMORZATO CON FORZANTE ESTERNA



si ottiene:



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Al variare dello smorzamento: