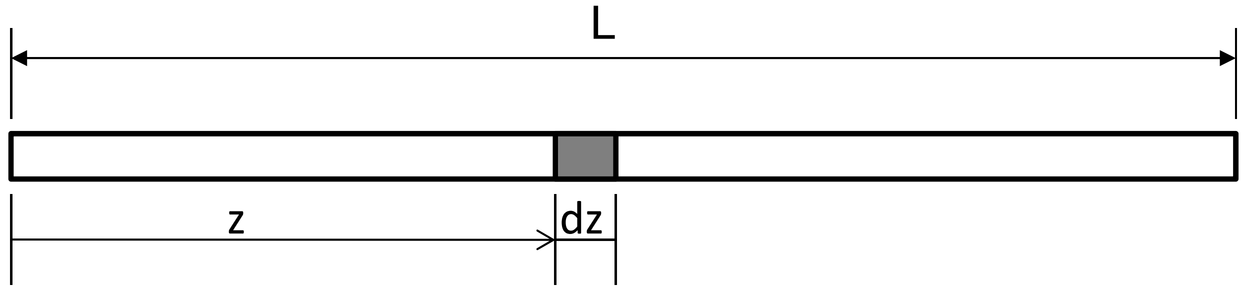
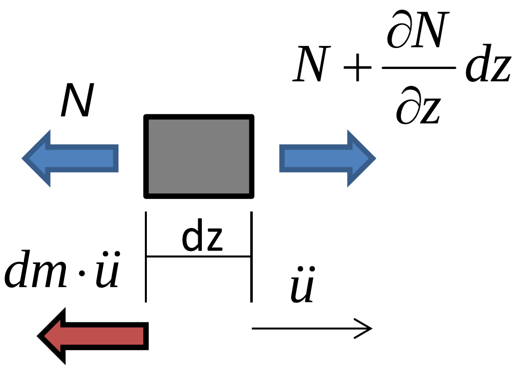
SISTEMI CONTINUI - ASPETTI GENERALI

- Un mezzo continuo ha **infiniti** gdl e, di conseguenza, **infiniti modi propri** di vibrare
- L'analisi delle vibrazioni di sistemi continui è molto complessa e sono disponibili soluzioni in forma chiusa (analitiche) solo per i casi più semplici
- Casi che saranno trattati
    - Trave
        - Vibrazioni estensionali (soluzione completa)
        - Vibrazioni flessionali (soluzione completa)
    - Piastra circolare
        - Vibrazioni flessionali (caratteristiche della soluzione)

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI





Equazione di equilibrio:

$$\rho A dz \cdot \ddot{u} = N + \frac{\partial N}{\partial z} dz - N$$

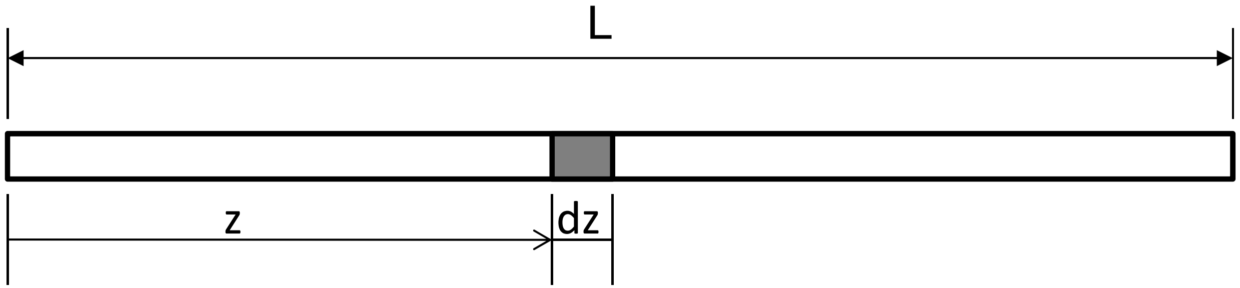
$$\rho A \ddot{u} = \frac{\partial N}{\partial z}$$

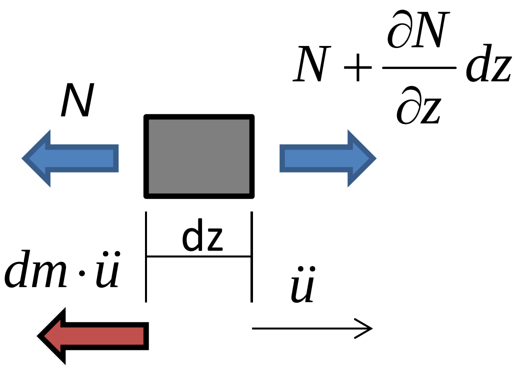
$$N = EA\varepsilon = EA \frac{\partial u}{\partial z}$$

$$\frac{\partial N}{\partial z} = EA \frac{\partial^2 u}{\partial z^2}$$

$$\boxed{\rho A \ddot{u} = EA \frac{\partial^2 u}{\partial z^2}}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



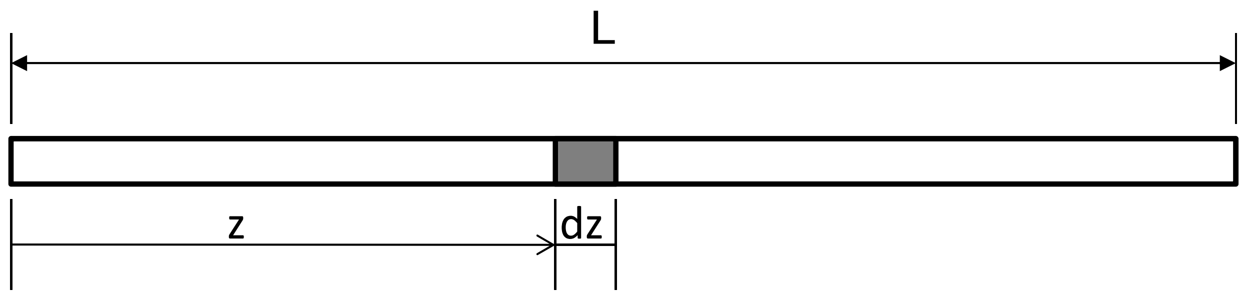


$$\rho \ddot{u} = E \frac{\partial^2 u}{\partial z^2}$$

$$\ddot{u} = \nu^2 \frac{\partial^2 u}{\partial z^2}$$

$$\nu = \sqrt{\frac{E}{\rho}}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



$$\ddot{u} = \nu^2 \frac{\partial^2 u}{\partial z^2}$$

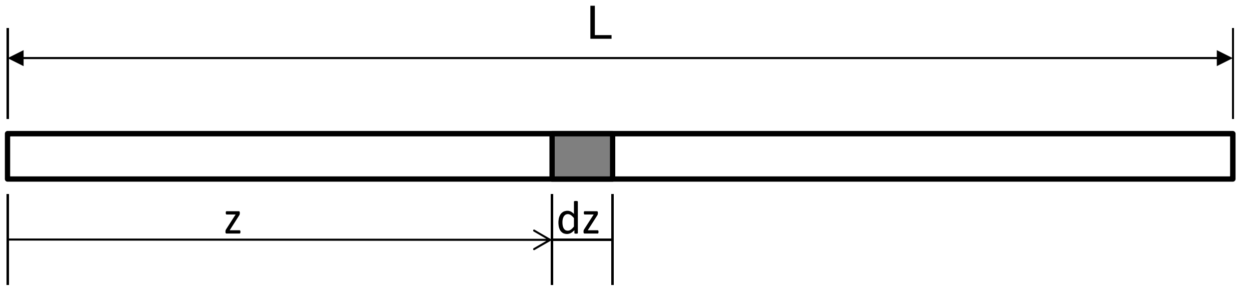
$$u(z,t) = Z(z)T(t)$$

$$\ddot{u}(z,t) = Z(z)\frac{\partial^2 T(t)}{\partial t^2}$$

$$\frac{\partial^2 u(z,t)}{\partial z^2} = T(t)\frac{\partial^2 Z(z)}{\partial z^2}$$

$$Z(z)\frac{\partial^2 T(t)}{\partial t^2} = \nu^2 T(t)\frac{\partial^2 Z(z)}{\partial z^2}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI

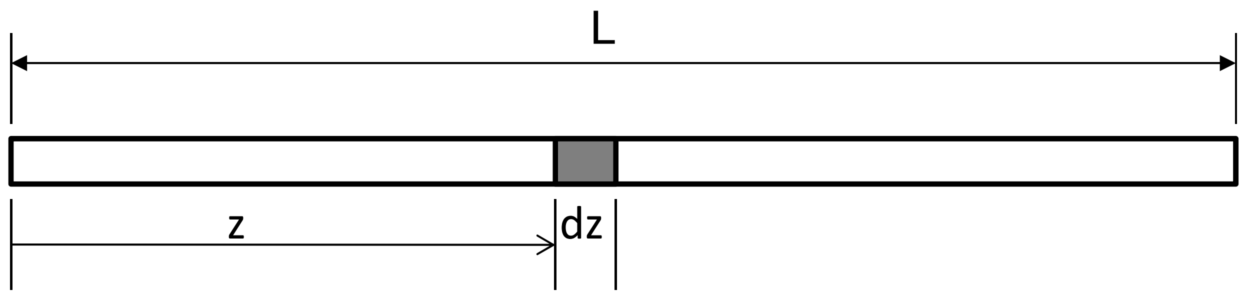


$$Z(z)\frac{\partial^2 T(t)}{\partial t^2} = v^2 T(t)\frac{\partial^2 Z(z)}{\partial z^2}$$

$$\frac{\dfrac{\partial^2 T(t)}{\partial t^2}}{T(t)} = v^2 \frac{\dfrac{\partial^2 Z(z)}{\partial z^2}}{Z(z)}$$

$$\frac{\ddot{T}}{T} = v^2 \frac{Z^{II}}{Z}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



$$\frac{\ddot{T}}{T} = a = v^2 \frac{Z^{II}}{Z}$$

$$\ddot{T} - aT = 0$$

$$v^2 Z^{II} - aZ = 0$$

$$a = -\omega^2$$

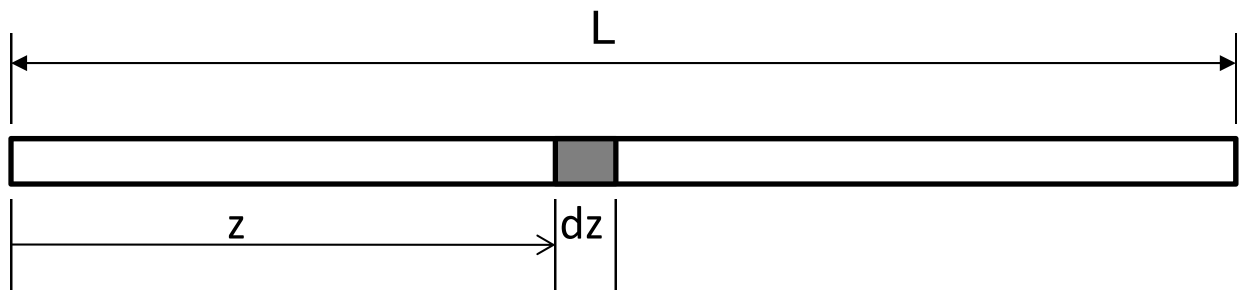
$$\ddot{T} + \omega^2 T = 0$$

$$T(t) = A \cdot \cos(\omega t) + B \cdot \sin(\omega t)$$

$$Z^{II} + \frac{\omega^2}{v^2} Z = 0$$

$$Z(z) = C \cdot \cos(\frac{\omega}{v} z) + D \cdot \sin(\frac{\omega}{v} z)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



Trave bloccata agli estremi

$$u(0,t) = Z(0)T(t) = 0 \qquad\qquad u(L,t) = Z(L)T(t) = 0$$

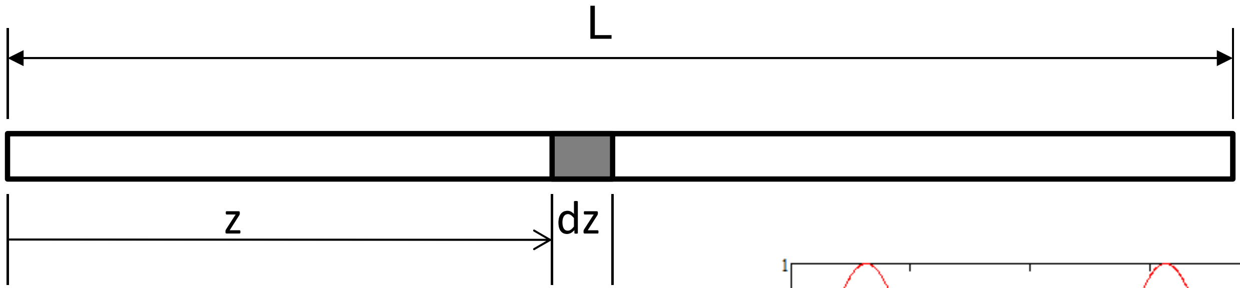
$$Z(0) = C \cdot \cos(\frac{\omega}{v} 0) + D \cdot \sin(\frac{\omega}{v} 0) = C = 0$$

$$Z(L) = D \cdot \sin(\frac{\omega}{v} L) = 0$$

$$\frac{\omega}{v} L = k\pi$$

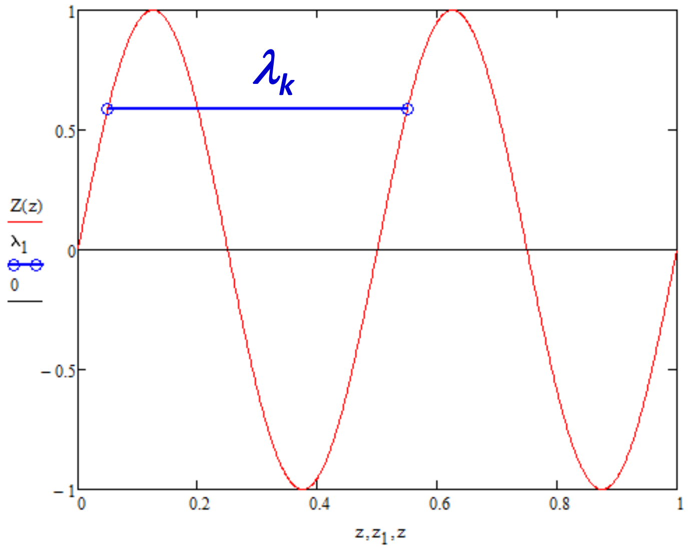
$$\omega = \frac{k\pi v}{L} = \frac{k\pi}{L}\sqrt{\frac{E}{\rho}}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



$$Z_k(z) = D_k \sin\left(k\frac{\pi}{L}z\right)$$

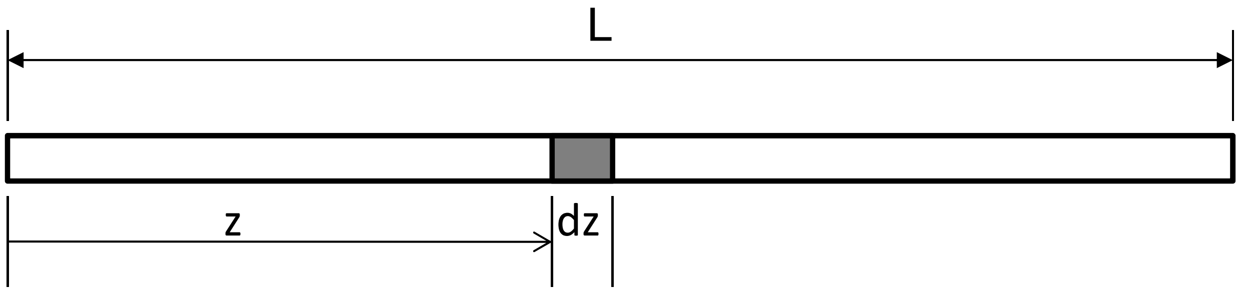
Si può introdurre il parametro: $\lambda_k = \dfrac{2L}{k}$



che corrisponde alla distanza tra punti corrispondenti in onde successive (lunghezza d’onda), esprimendo la funzione normale come:

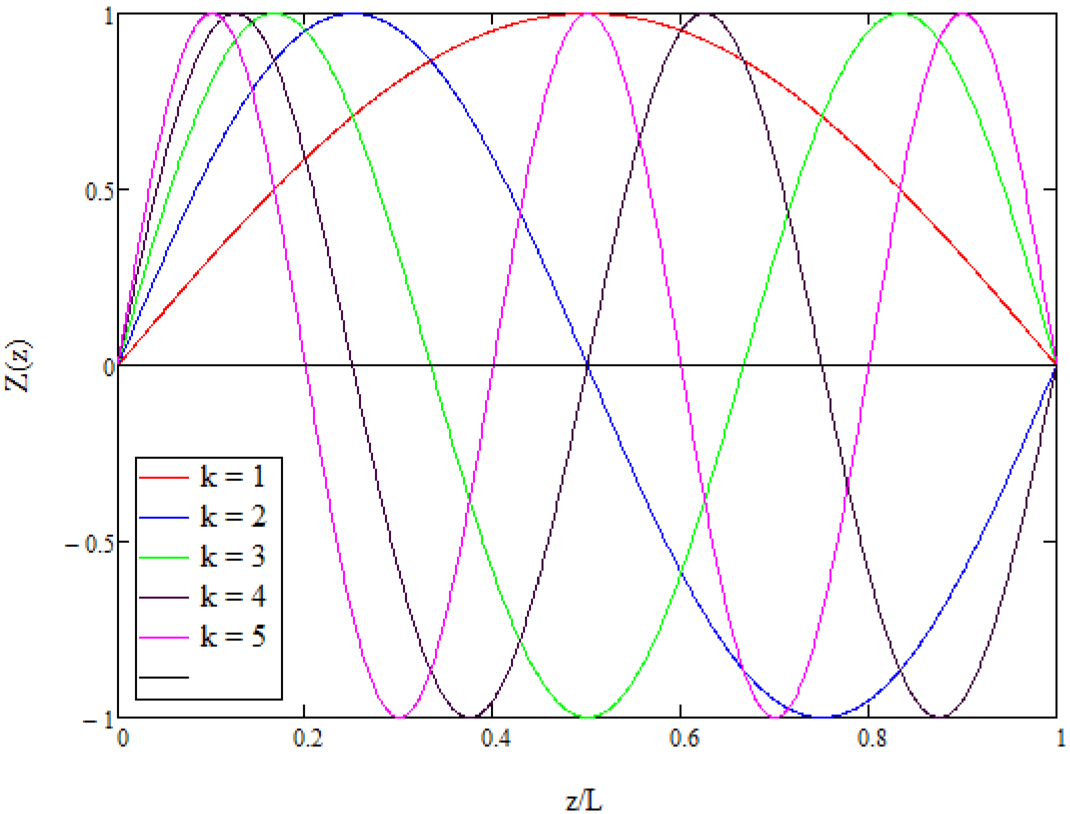
$$Z_k(z) = D_k \sin\left(\frac{2\pi}{\lambda_k}z\right)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI

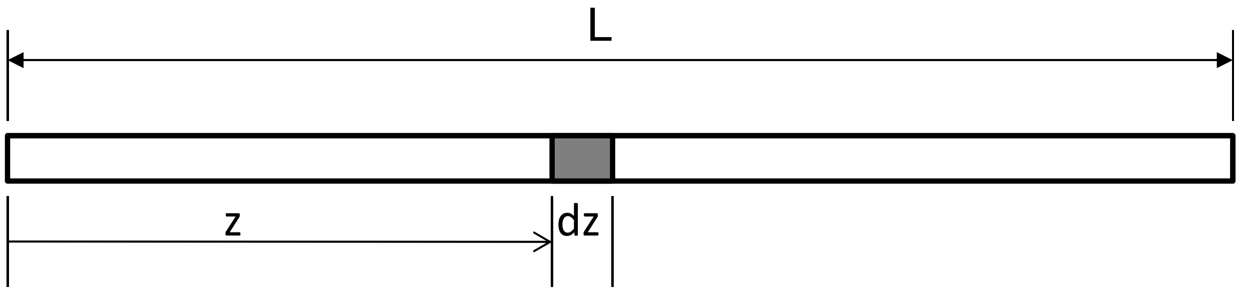


$$\lambda_k = \frac{2L}{k}$$

$$Z_k(z) = D_k \sin\left(\frac{2\pi}{\lambda_k} z\right)$$



SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



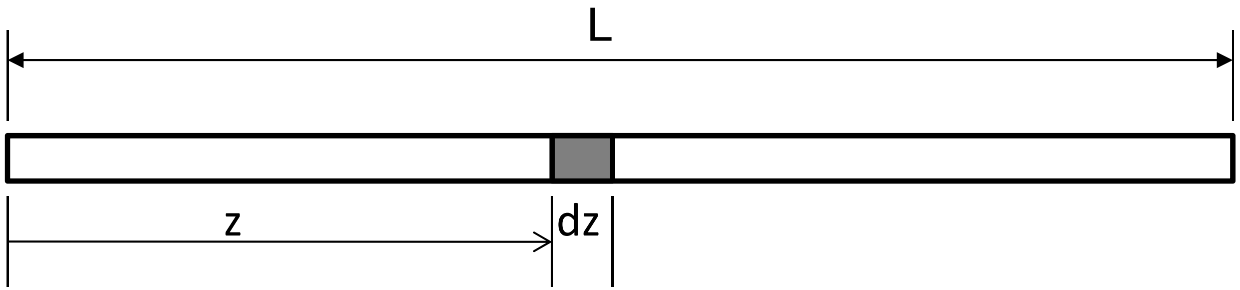
L'oscillazione libera secondo la generica forma modale sarà data da una funzione del tipo:

$$u_k(z,t) = \left[A_k D_k \cdot \cos(\omega_k t) + B_k D_k \cdot \sin(\omega_k t)\right] \sin(\frac{\omega_k}{v} z)$$

nella quale le due costanti $A_k D_k$ e $B_k D_k$ devono essere determinate in base alle condizioni inziali. L'oscillazione libera generale sarà infine data da:

$$u(z,t) = \sum_{k=1}^{\infty} u_k(z,t) = \sum_{k=1}^{\infty} \left[A_k D_k \cdot \cos(\omega_k t) + B_k D_k \cdot \sin(\omega_k t)\right] \sin(\frac{\omega_k}{v} z)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



L’oscillazione libera secondo la generica forma modale sarà data da una funzione del tipo:

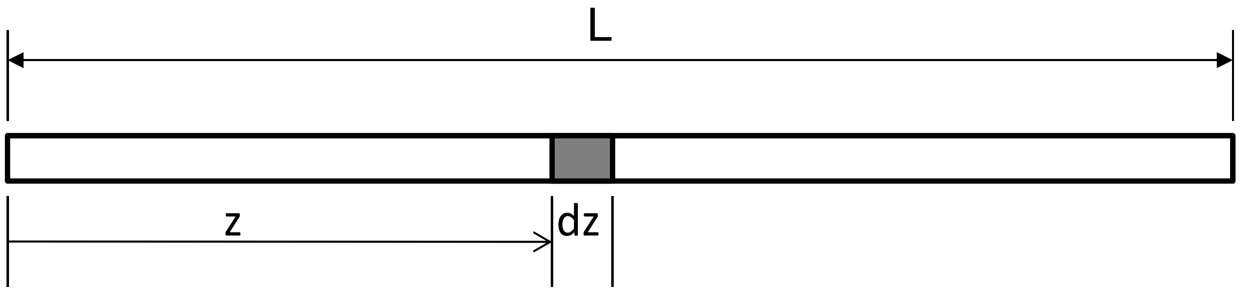
$$u_k(z,t) = E_k \cdot \cos(\omega_k t)\sin(\frac{\omega_k}{v}z) + F_k \cdot \sin(\omega_k t)\sin(\frac{\omega_k}{v}z)$$

$$u_k(z,t) = \frac{E_k}{2}\cdot\left[\cos(\omega_k t)\sin(\frac{\omega_k}{v}z) + \sin(\omega_k t)\cos(\frac{\omega_k}{v}z) - \sin(\omega_k t)\cos(\frac{\omega_k}{v}z) + \cos(\omega_k t)\sin(\frac{\omega_k}{v}z)\right]$$

$$+\frac{F_k}{2}\cdot\left[\sin(\omega_k t)\sin(\frac{\omega_k}{v}z) + \cos(\omega_k t)\cos(\frac{\omega_k}{v}z) - \cos(\omega_k t)\cos(\frac{\omega_k}{v}z) + \sin(\omega_k t)\sin(\frac{\omega_k}{v}z)\right]$$

$$u_k(z,t) = \frac{E_k}{2}\cdot\left[\sin(\frac{\omega_k}{v}z + \omega_k t) + \sin(\frac{\omega_k}{v}z - \omega_k t)\right] + \frac{F_k}{2}\cdot\left[\cos(\frac{\omega_k}{v}z + \omega_k t) + \cos(\frac{\omega_k}{v}z - \omega_k t)\right]$$

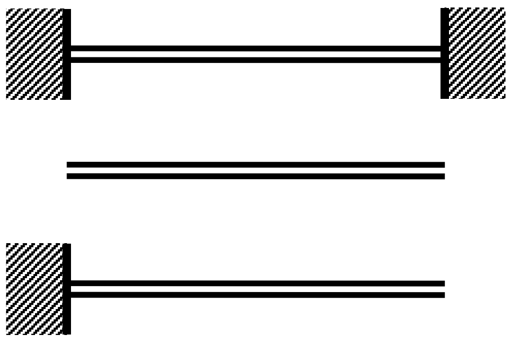
SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI ESTENSIONALI



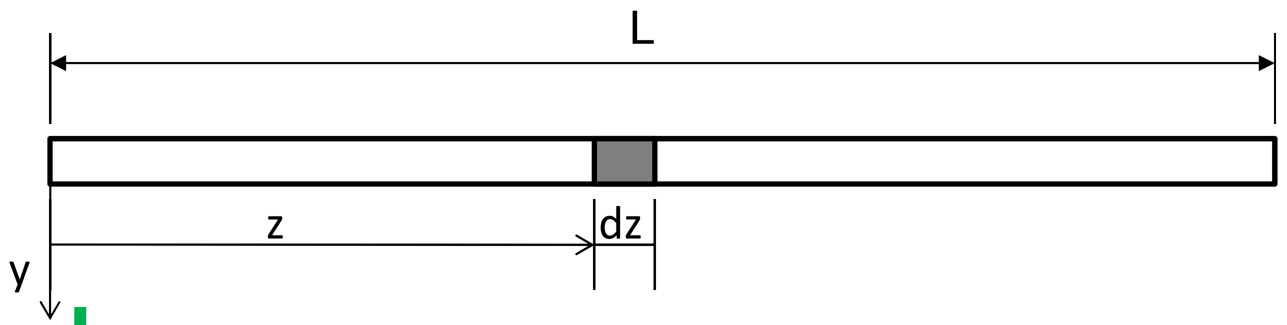
Le frequenze proprie, per altre modalità di vincolo, sono date dalla seguente relazione generale, nella quale il coefficiente K(k) dipende da queste ultime:

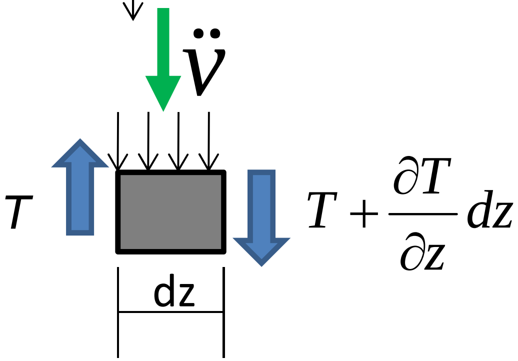
$$\omega_k = K(k)\frac{\pi}{L}\sqrt{\frac{E}{\rho}} \qquad k = 1,2,3\ldots$$

	K(k)
Incastrato-incastrato	k
Libero-libero	k
Incastrato-libero	k+1/2



SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI





Equazione di equilibrio:

$$\rho A dz \cdot \ddot{v} = T + \frac{\partial T}{\partial z} dz - T$$

$$\rho A \ddot{v} = \frac{\partial T}{\partial z}$$

$$T = -EJv^{III}$$

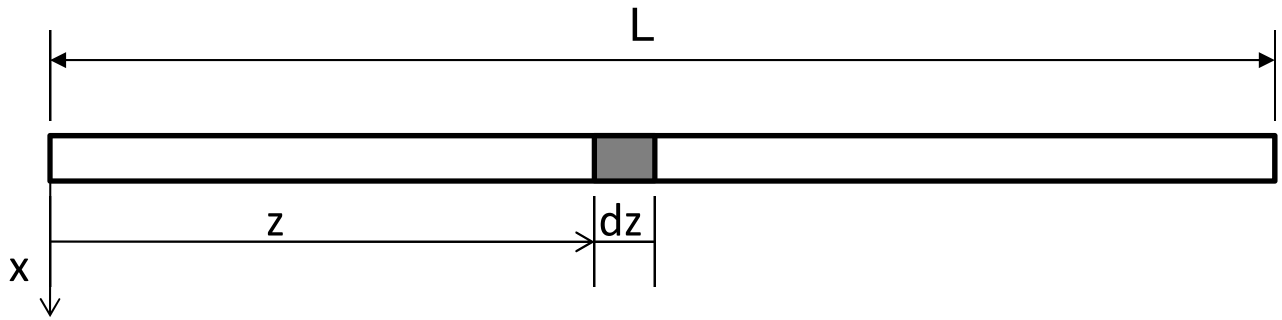
$$\frac{\partial T}{\partial z} = -EJv^{IV}$$

$$\rho A \ddot{v} = -EJv^{IV}$$

$$\ddot{v} = -k^2 v^{IV}$$

$$k = \sqrt{\frac{EJ}{\rho A}}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



$$\ddot{v} = k^2 v^{IV}$$

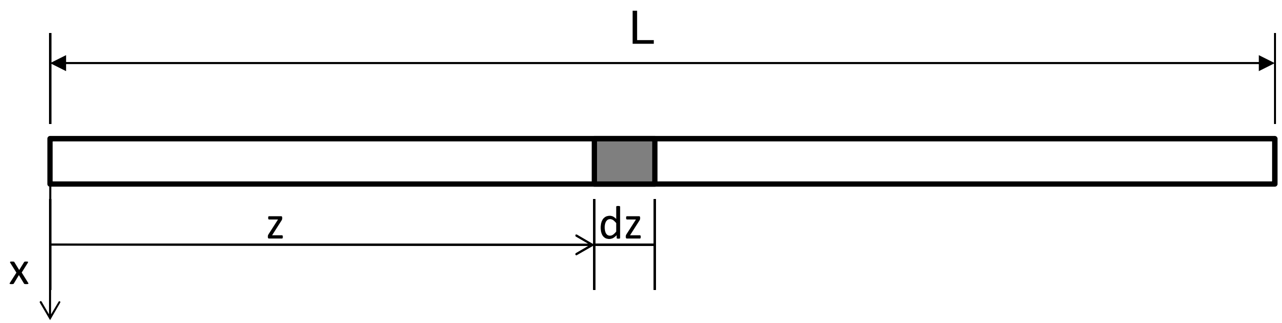
$$v(z,t) = V(z)T(t)$$

$$\frac{\ddot{T}}{T} = -k^2 \frac{V^{IV}}{V} = -\omega^2$$

$$\ddot{T} + \omega^2 T = 0$$

$$V^{IV} - \frac{\omega^2}{k^2} V = 0$$

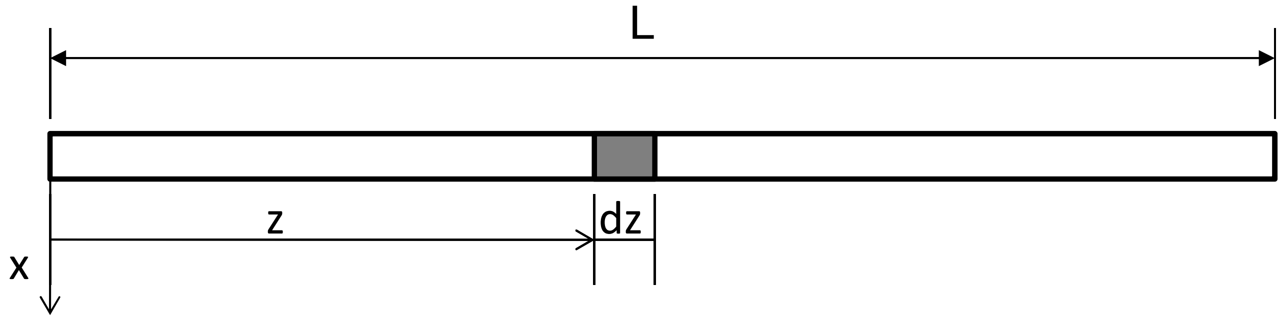
SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



$$\ddot{T} + \omega^2 T = 0$$

$$T(t) = A \cdot \cos(\omega t) + B \cdot \sin(\omega t)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



$$V^{IV} - \frac{\omega^2}{k^2} V = 0$$

$$\chi = \sqrt{\frac{\omega}{k}}$$

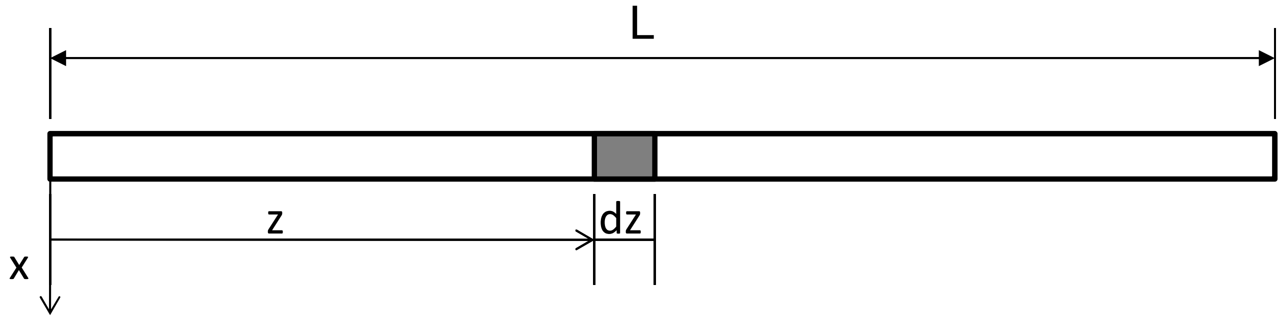
$$V^{IV} - \chi^4 V = 0$$

$$s^4 - \chi^4 = 0$$

$$s = \pm\chi; \pm i\chi$$

$$V(z) = C_1 \cdot e^{\chi z} + D_1 \cdot e^{-\chi z} + E_1 \cdot e^{i\chi z} + F_1 \cdot e^{-i\chi z}$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



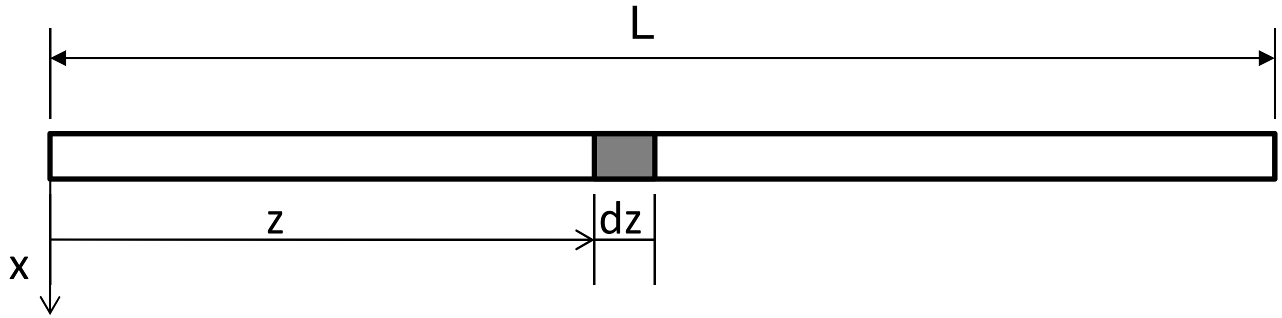
$$V(z) = C_1 \cdot e^{\chi z} + D_1 \cdot e^{-\chi z} + E_1 \cdot e^{i\chi z} + F_1 \cdot e^{-i\chi z}$$

$$\sin(\chi z) = \frac{e^{i\chi z} - e^{-i\chi z}}{2i} \qquad \sinh(\chi z) = \frac{e^{\chi z} - e^{-i\chi z}}{2}$$

$$\cos(\chi z) = \frac{e^{i\chi z} + e^{-i\chi z}}{2} \qquad \cosh(\chi z) = \frac{e^{i\chi z} + e^{-i\chi z}}{2}$$

$$V(z) = C \cdot \sin(\chi z) + D \cdot \cos(\chi z) + E \cdot \sinh(\chi z) + F_1 \cdot \cosh(\chi z)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



$$\sinh(0)=0 \quad \frac{d}{dt}\sinh(t)=\cosh(t) \quad (\cosh(t))^2-(\sinh(t))^2=1$$

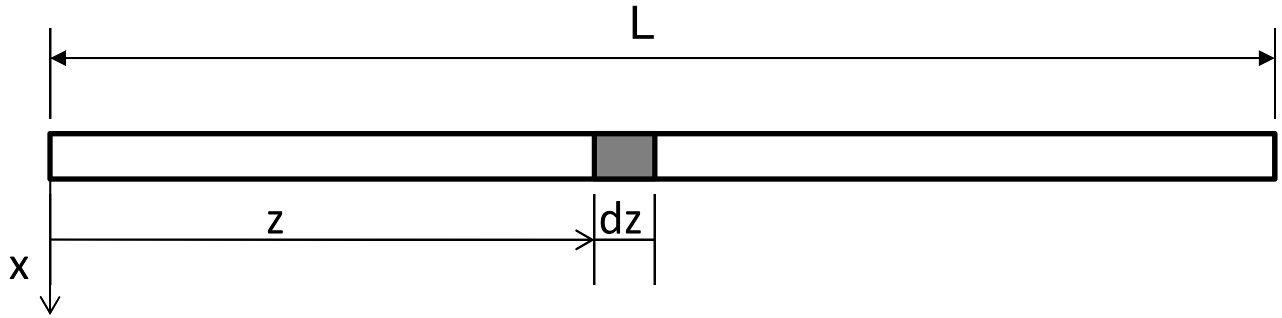
$$\cosh(0)=1 \quad \frac{d}{dt}\cosh(t)=\sinh(t)$$

$$V^{I}(z)=C\chi\cos(\chi z)-D\chi\sin(\chi z)+E\chi\cosh(\chi z)+F\chi\sinh(\chi z)$$

$$V^{II}(z)=-C\chi^2\sin(\chi z)-D\chi^2\cos(\chi z)+E\chi^2\sinh(\chi z)+F\chi^2\cosh(\chi z)$$

$$V^{III}(z)=-C\chi^3\cos(\chi z)+D\chi^3\sin(\chi z)+E\chi^3\cosh(\chi z)+F\chi^3\sinh(\chi z)$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave libera agli estremi (z=0, L)

$$V^{II}(0) = V^{III}(0) = V^{II}(L) = V^{III}(L) = 0$$

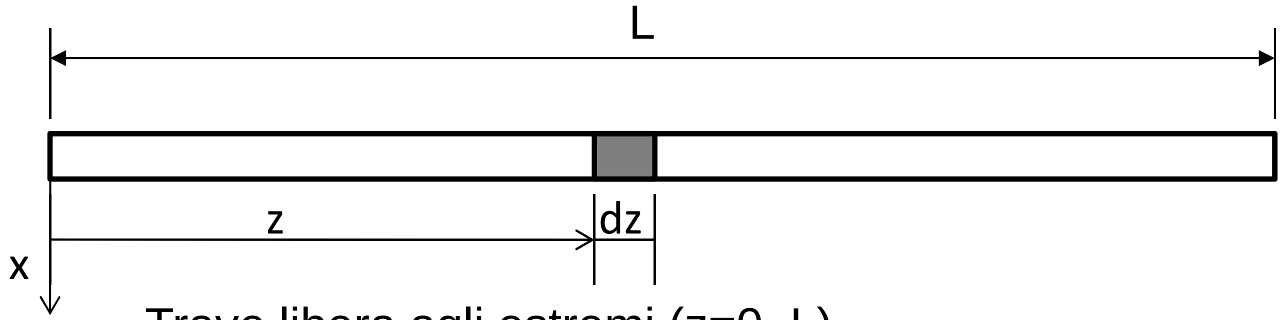
$$V^{II}(0) = -C\chi^2 \sin(0) - D\chi^2 \cos(0) + E\chi^2 \sinh(0) + F\chi^2 \cosh(0) = F - D = 0$$

$$V^{III}(0) = -C\chi^3 \cos(0) + D\chi^3 \sin(0) + E\chi^3 \cosh(0) + F\chi^3 \sinh(0) = E - C = 0$$

$$V^{II}(L) = -C\chi^2 \sin(\chi L) - D\chi^2 \cos(\chi L) + E\chi^2 \sinh(\chi L) + F\chi^2 \cosh(\chi L) = 0$$

$$V^{III}(L) = -C\chi^3 \cos(\chi L) + D\chi^3 \sin(\chi L) + E\chi^3 \cosh(\chi L) + F\chi^3 \sinh(\chi L) = 0$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave libera agli estremi (z=0, L)

$$F - D = 0$$

$$E - C = 0$$

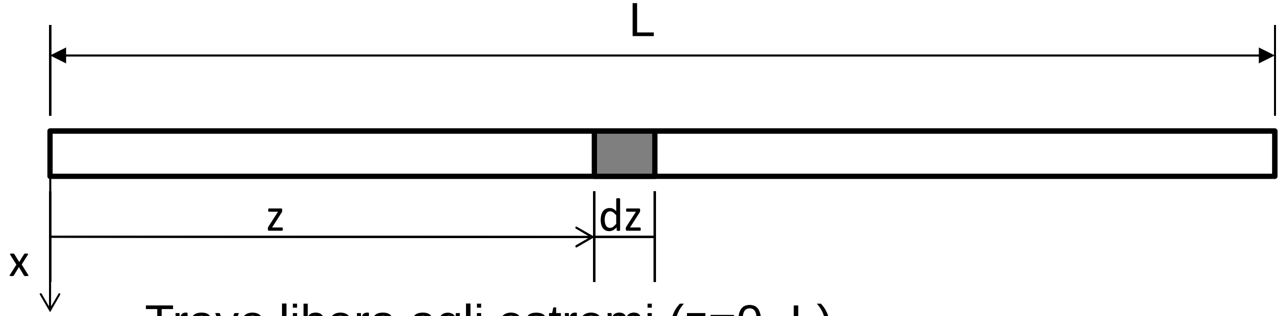
$$-C\chi^2 \sin(\chi L) - D\chi^2 \cos(\chi L) + E\chi^2 \sinh(\chi L) + F\chi^2 \cosh(\chi L) = 0$$

$$-C\chi^3 \cos(\chi L) + D\chi^3 \sin(\chi L) + E\chi^3 \cosh(\chi L) + F\chi^3 \sinh(\chi L) = 0$$

Per avere sln. non banale:

$$\det\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -\chi^2 \sin(\chi L) & -\chi^2 \cos(\chi L) & \chi^2 \sinh(\chi L) & \chi^2 \cosh(\chi L) \\ -\chi^3 \cos(\chi L) & \chi^3 \sin(\chi L) & \chi^3 \cosh(\chi L) & \chi^3 \sinh(\chi L) \end{bmatrix} = 0$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave libera agli estremi (z=0, L)

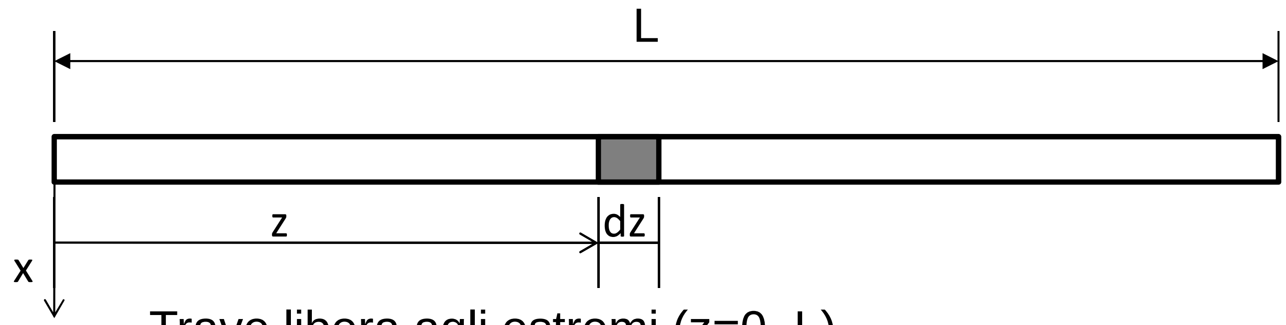
$$\det\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -\chi^2\sin(\chi L) & -\chi^2\cos(\chi L) & \chi^2\sinh(\chi L) & \chi^2\cosh(\chi L) \\ -\chi^3\cos(\chi L) & \chi^3\sin(\chi L) & \chi^3\cosh(\chi L) & \chi^3\sinh(\chi L) \end{bmatrix} = 0$$

$$\left\{\begin{matrix} -\left[\chi^2\sinh(\chi L)\chi^3\sinh(\chi L) - \chi^2\cosh(\chi L)\chi^3\cosh(\chi L)\right] - \\ \left[-\chi^2\sin(\chi L)\chi^3\sinh(\chi L) + \chi^2\cosh(\chi L)\chi^3\cos(\chi L)\right] \end{matrix}\right\} -$$

$$-\left\{\begin{matrix} -\left[-\chi^2\cos(\chi L)\chi^3\cosh(\chi L) - \chi^2\sinh(\chi L)\chi^3\sin(\chi L)\right] + \\ \left[-\chi^2\sin(\chi L)\chi^3\sin(\chi L) - \chi^2\cos(\chi L)\chi^3\cos(\chi L)\right] \end{matrix}\right\} =$$

SISTEMI CONTINUI

TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave libera agli estremi (z=0, L)

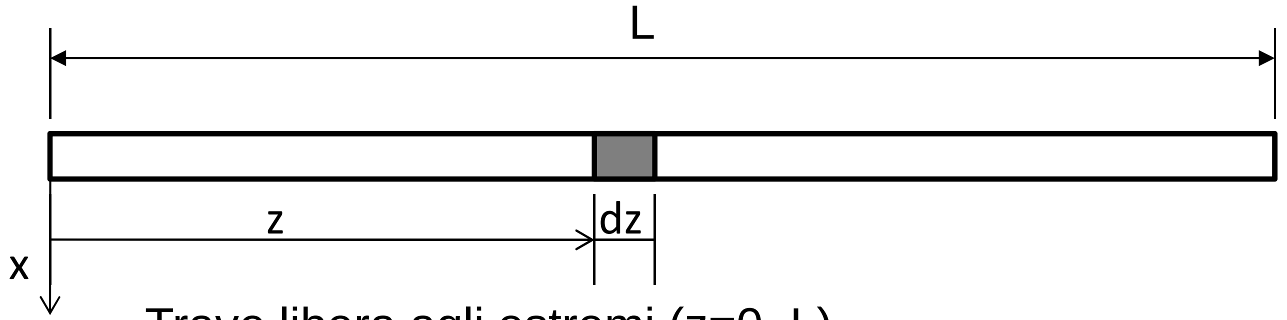
$$\begin{Bmatrix} -\left[\chi^5 \sinh^2(\chi L) - \chi^5 \cosh^2(\chi L)\right] - \\ \left[-\chi^5 \sin(\chi L)\sinh(\chi L) + \chi^5 \cosh(\chi L)\cos(\chi L)\right] \end{Bmatrix} -$$

$$= -\begin{Bmatrix} \left[\chi^5 \cos(\chi L)\cosh(\chi L) + \chi^5 \sinh(\chi L)\sin(\chi L)\right] + \\ \left[-\chi^5 \sin^2(\chi L) - \chi^5 \cos^2(\chi L)\right] \end{Bmatrix} =$$

$$= \begin{Bmatrix} \chi^5\left[\cosh^2(\chi L) - \sinh^2(\chi L)\right] + \\ \chi^5\left[\sin(\chi L)\sinh(\chi L) - \cosh(\chi L)\cos(\chi L)\right] \end{Bmatrix} -$$

$$- \begin{Bmatrix} \chi^5\left[\cos(\chi L)\cosh(\chi L) + \sinh(\chi L)\sin(\chi L)\right] - \\ \chi^5\left[\sin^2(\chi L) + \cos^2(\chi L)\right] \end{Bmatrix} =$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



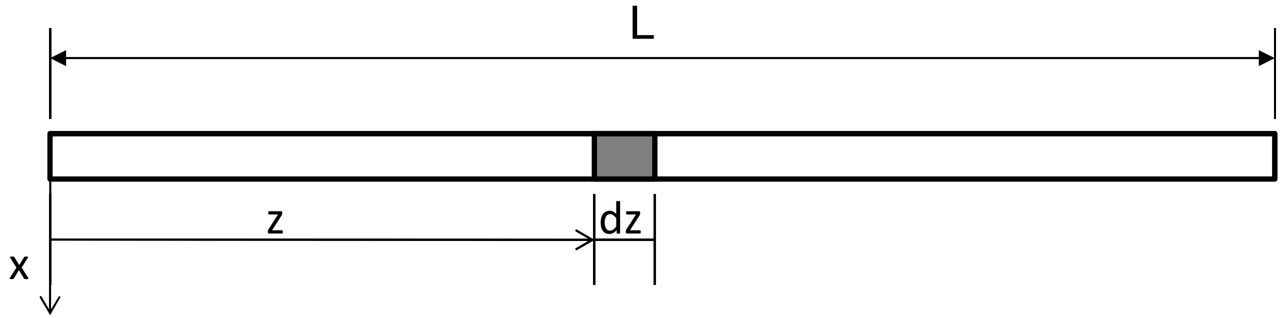
Trave libera agli estremi (z=0, L)

$$= \chi^5 \{1 + \sin(\chi L)\sinh(\chi L) - \cosh(\chi L)\cos(\chi L) - \cos(\chi L)\cosh(\chi L) - \sinh(\chi L)\sin(\chi L) + 1\} =$$

$$= 2\chi^5 \{1 - \cosh(\chi L)\cos(\chi L)\} = 0$$

$$\cosh(\chi L)\cos(\chi L) = 1$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave libera agli estremi (z=0, L)

$$\cosh(\chi_n L)\cos(\chi_n L) = 1$$

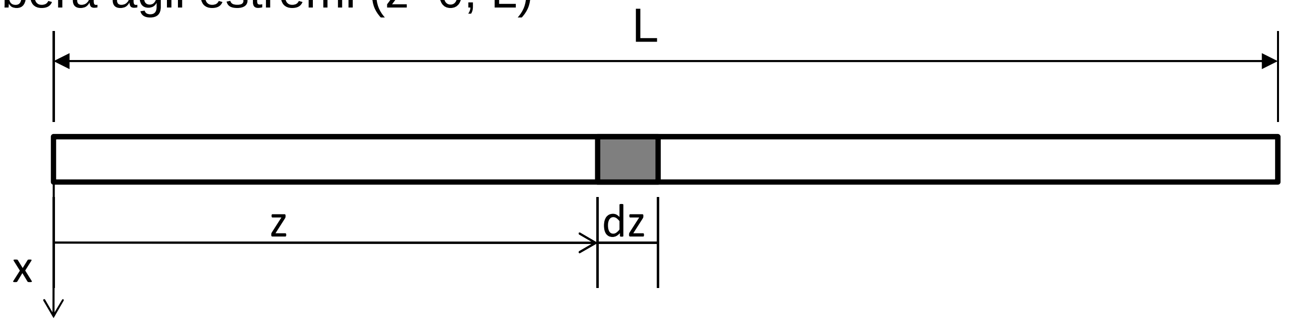
$$\omega_n = \chi_n^2 k = \chi_n^2 \sqrt{\frac{EJ}{\rho A}} = (\chi_n L)^2 \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}$$

n	1	2	3	4	>4
$\chi_n L$	4.730	7.853	10.996	14.137	$(n+1/2)\pi$

Valore asintotico (valido dopo i primi 3-4 termini)

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI

Trave libera agli estremi (z=0, L)



$$F = D$$

$$E = C$$

$$-C\sin(\chi_n L) - D\cos(\chi_n L) + C\sinh(\chi_n L) + D\cosh(\chi_n L) = 0$$

$$C[\sinh(\chi_n L) - \sin(\chi_n L)] - D[\cos(\chi_n L) - \cosh(\chi_n L)] = 0$$

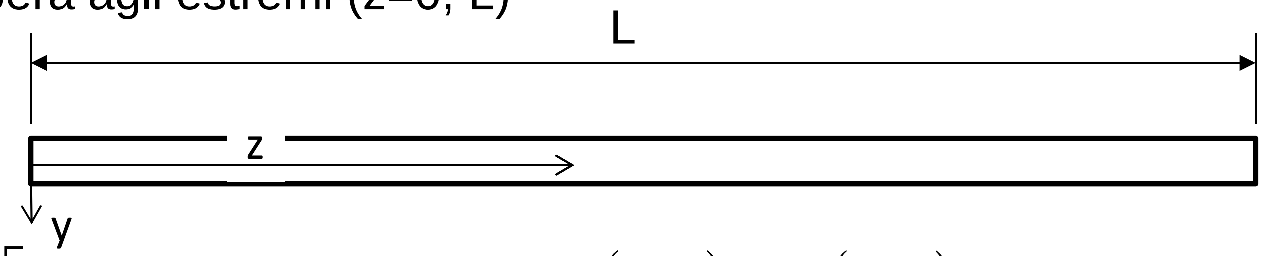
$$D = C\frac{[\sinh(\chi_n L) - \sin(\chi_n L)]}{[\cos(\chi_n L) - \cosh(\chi_n L)]}$$

$$F = C\frac{[\sinh(\chi_n L) - \sin(\chi_n L)]}{[\cos(\chi_n L) - \cosh(\chi_n L)]}$$

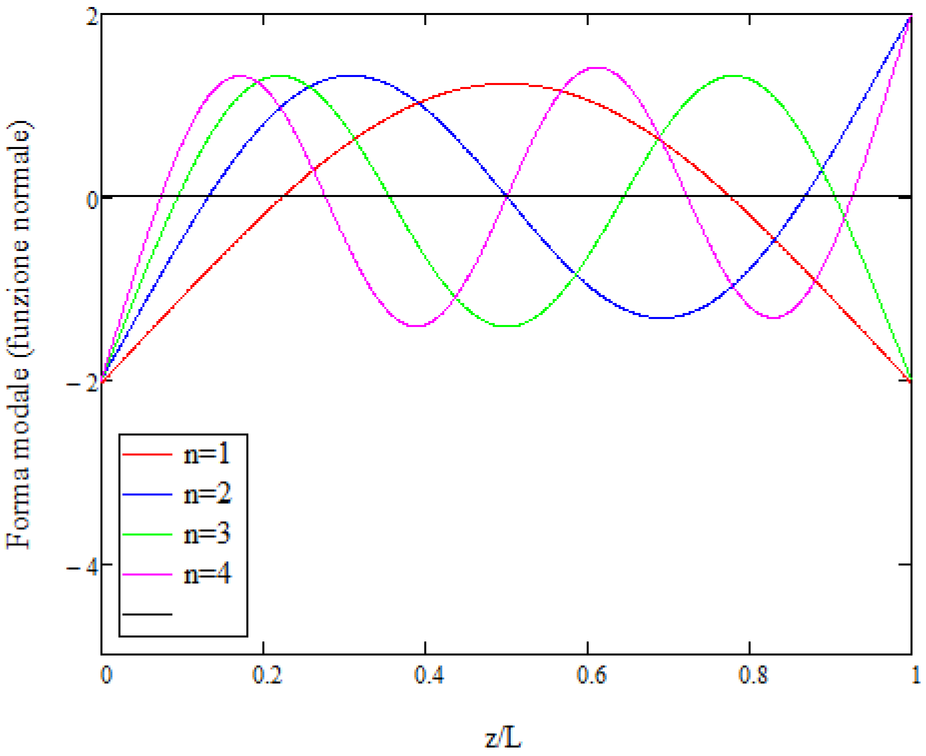
$$E = C$$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI

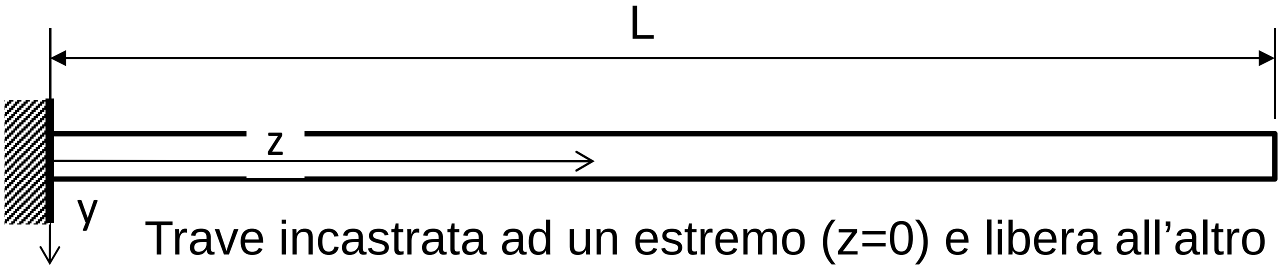
Trave libera agli estremi (z=0, L)



$$V_n(z) = C_n\left[\sin(\chi_n z) + \sinh(\chi_n z) + \frac{\sinh(\chi_n L) - \sin(\chi_n L)}{\cos(\chi_n L) - \cosh(\chi_n L)}\left(\cos(\chi_n z) + \cosh(\chi_n z)\right)\right]$$



SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



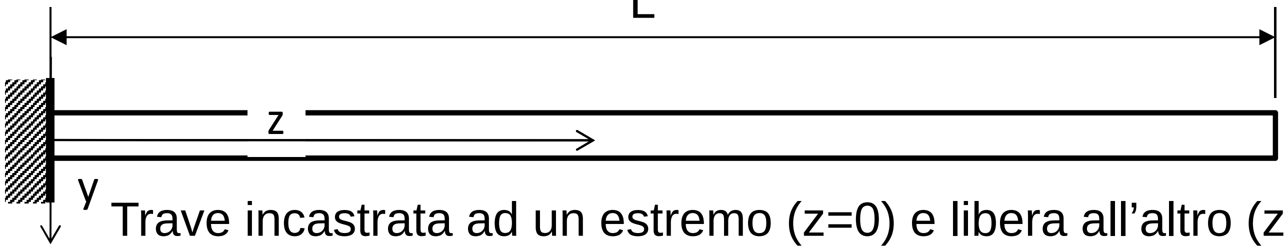
Trave incastrata ad un estremo (z=0) e libera all'altro (z=L)

Condizioni al contorno $V(0) = V^{I}(0) = 0 \qquad V^{II}(L) = V^{III}(L) = 0$

$$\omega_n = (\chi_n L)^2 \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}$$

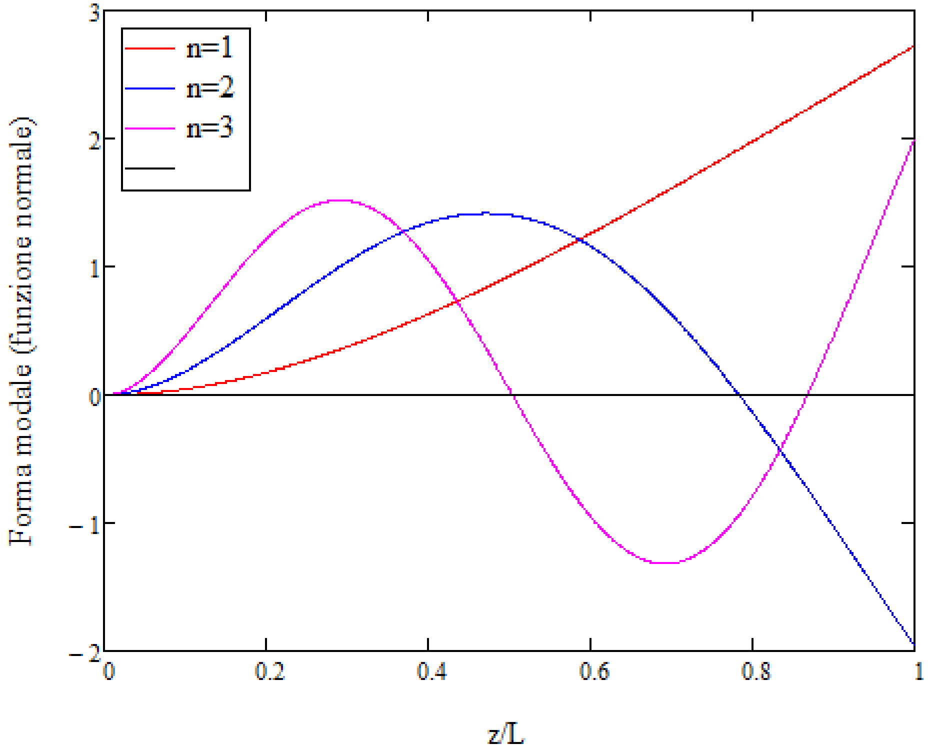
n	1	2	3	4	>4
$\chi_n L$	1.875	4.694	7.855	10.996	(n-1/2)π

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI

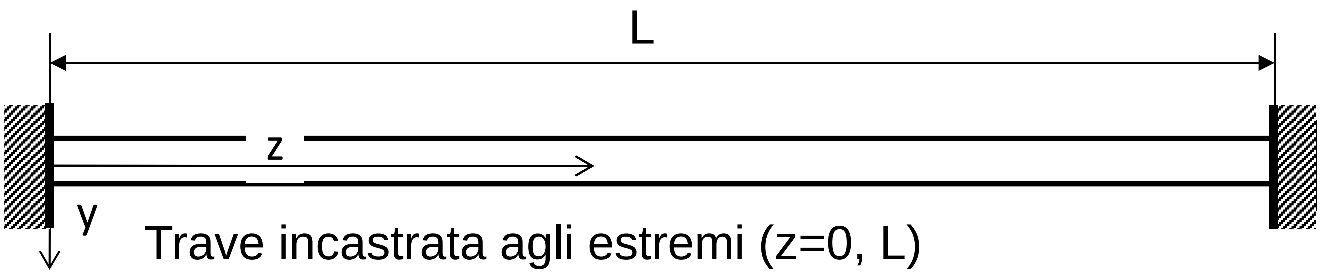


Trave incastrata ad un estremo (z=0) e libera all'altro (z=L)

$$V_n(z) = C_n\left[\sin(\chi_n z) - \sinh(\chi_n z) - \frac{\sinh(\chi_n L) + \sin(\chi_n L)}{\cos(\chi_n L) + \cosh(\chi_n L)}\left(\cos(\chi_n z) - \cosh(\chi_n z)\right)\right]$$



SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



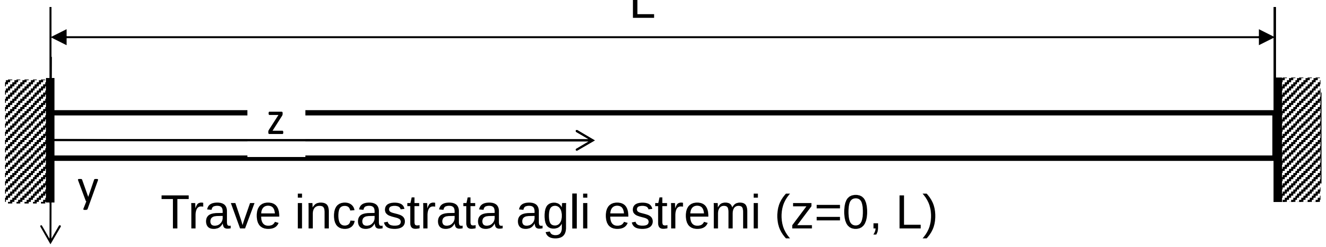
Trave incastrata agli estremi (z=0, L)

Condizioni al contorno $V(0) = V^{I}(0) = 0$ $\quad V^{II}(L) = V^{III}(L) = 0$

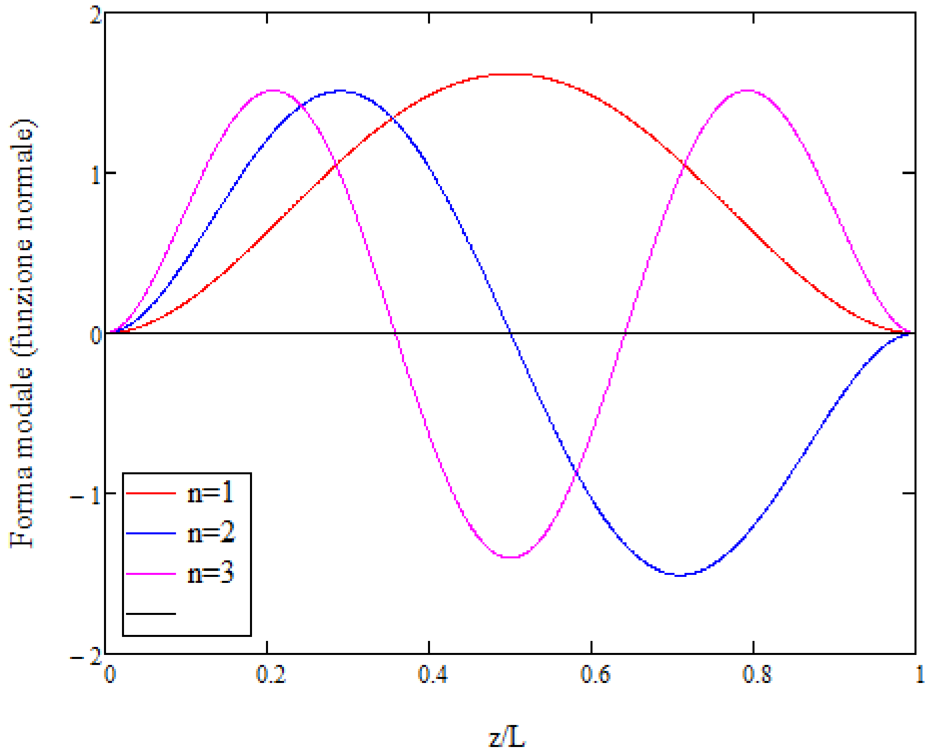
$$\omega_n = (\chi_n L)^2 \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}$$

n	1	2	3	4	>4
$\chi_n L$	4.730	7.853	10.996	14.137	$(n+1/2)\pi$

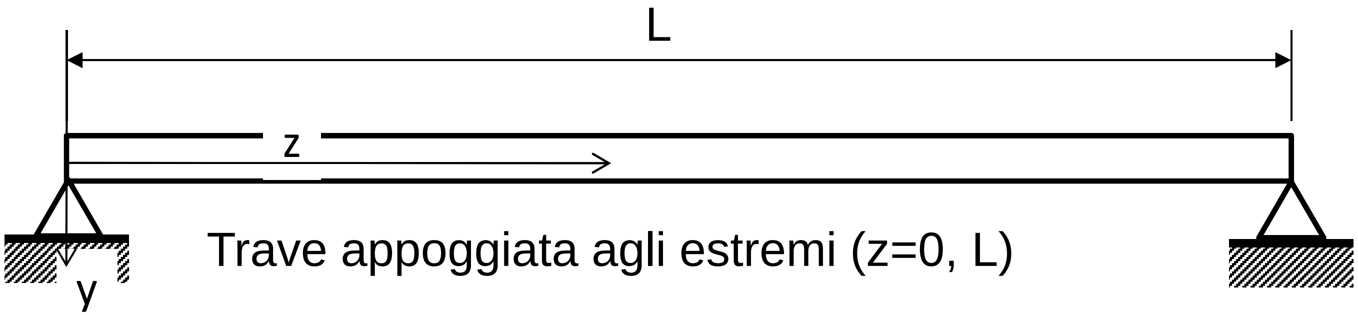
SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



$$V_n(z) = C_n\left[\sinh(\chi_n z) - \sin(\chi_n z) + \frac{\sinh(\chi_n L) - \sin(\chi_n L)}{\cos(\chi_n L) - \cosh(\chi_n L)}\left(\cosh(\chi_n z) - \cos(\chi_n z)\right)\right]$$



SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



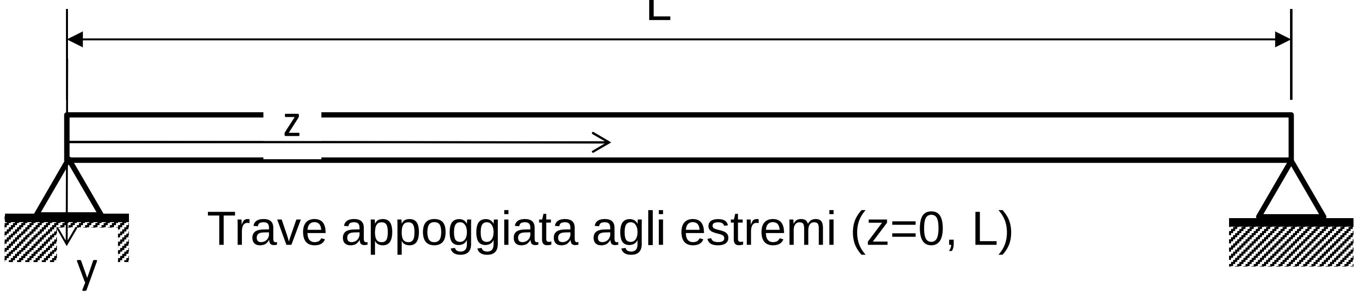
Trave appoggiata agli estremi (z=0, L)

Condizioni al contorno $V(0) = V^{II}(0) = 0 \qquad V(L) = V^{II}(L) = 0$

$$\omega_n = (\chi_n L)^2 \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}$$

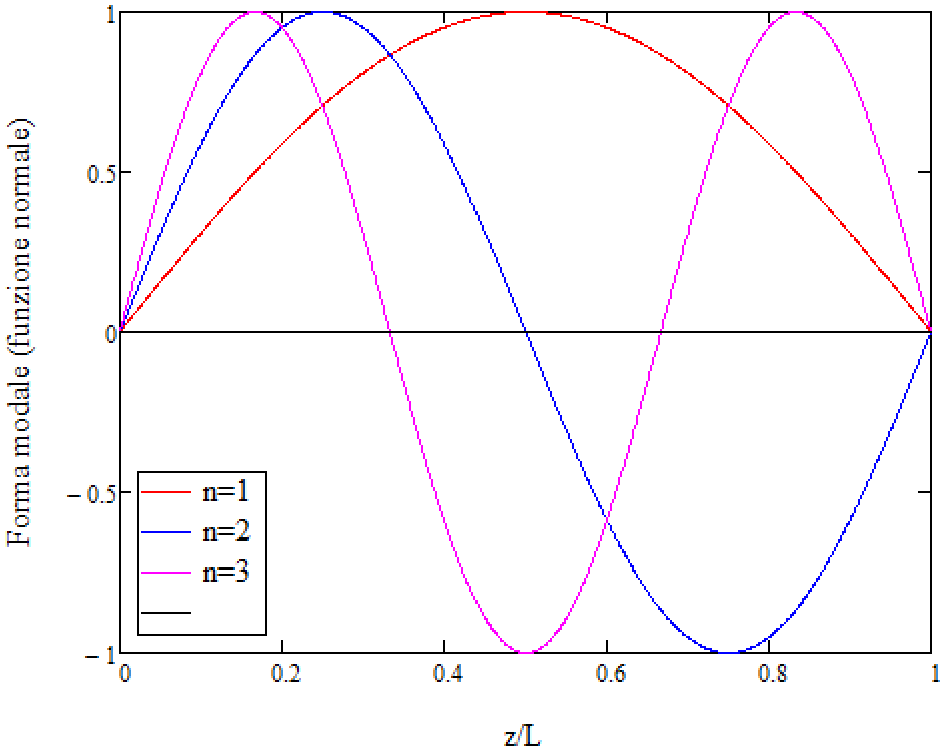
n	1	2	3	4	>4
$\chi_n L$	π	2π	3π	4π	$n\pi$

SISTEMI CONTINUI
TRAVE SOGGETTA A VIBRAZIONI FLESSIONALI



Trave appoggiata agli estremi (z=0, L)

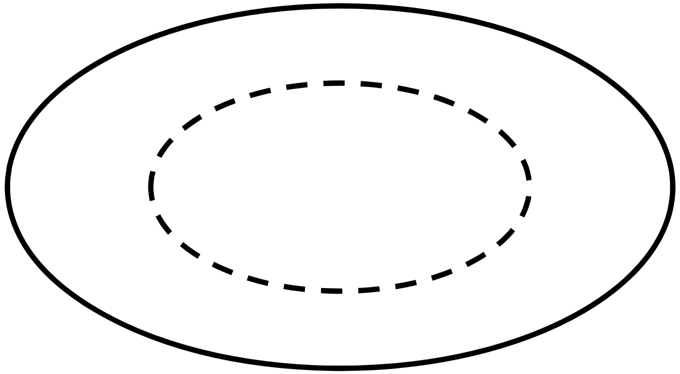
$$V_n(z) = C_n\left[\sin\left(\chi_n z\right)\right]$$

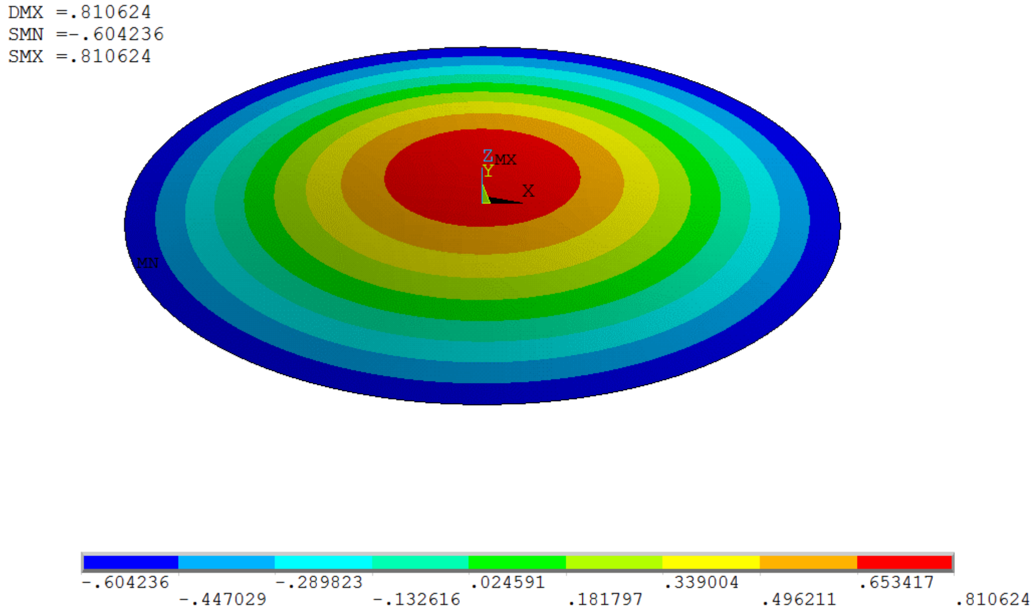


SISTEMI CONTINUI
PIASTRA CIRCOLARE

Le vibrazioni proprie di una piastra circolare possono prevedere:

- forme modali simmetriche, che presentano circonferenze nodali, i cui punti non si spostano durante la vibrazione

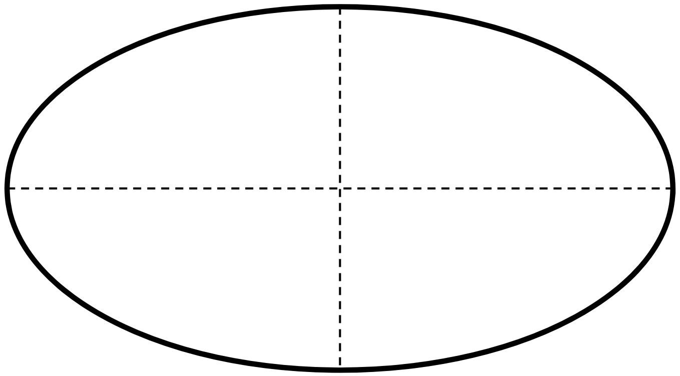


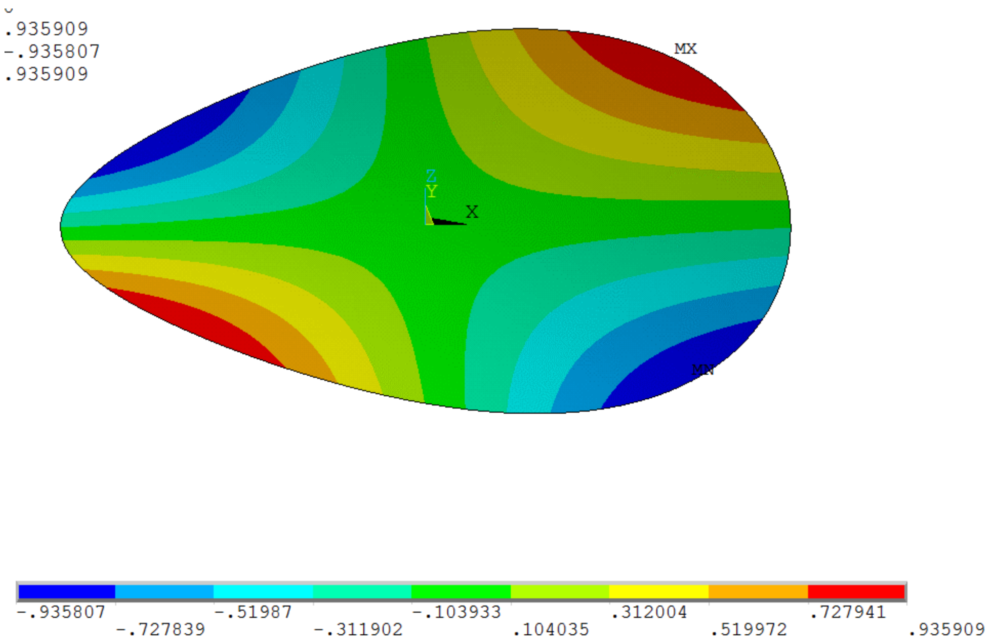


SISTEMI CONTINUI
PIASTRA CIRCOLARE

Le vibrazioni proprie di una piastra circolare possono prevedere:

- **forme modali simmetriche**, che presentano **circonferenze nodali**, i cui punti non si spostano durante la vibrazione
- **forme modali anti-simmetriche**, che presentano **diametri nodali**, i cui punti non si spostano durante la vibrazione

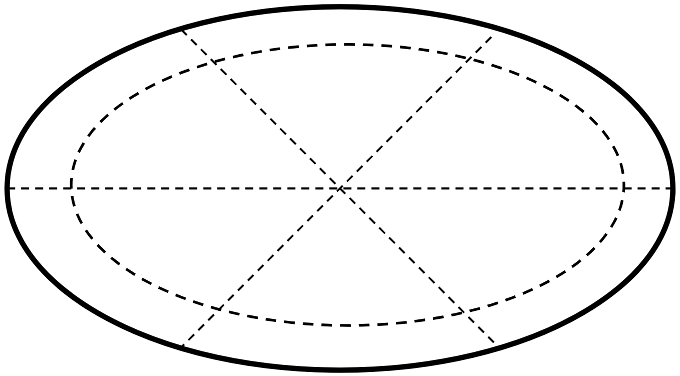


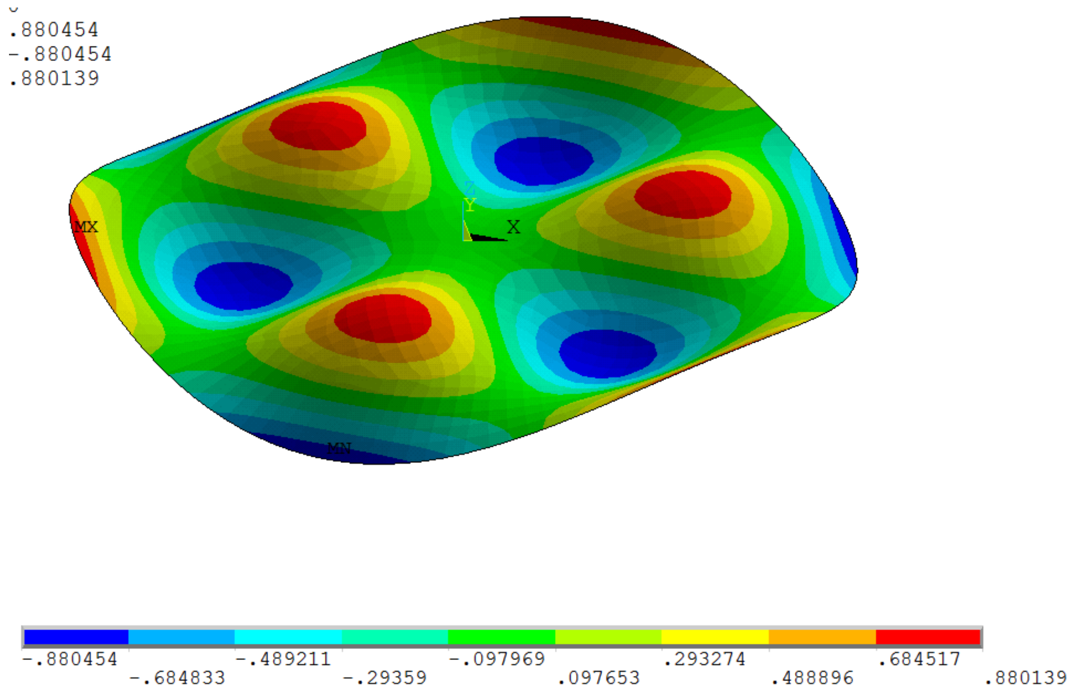


SISTEMI CONTINUI
PIASTRA CIRCOLARE

Le vibrazioni proprie di una piastra circolare possono prevedere:

- **forme modali simmetriche**, che presentano **circonferenze nodali**, i cui punti non si spostano durante la vibrazione
- **forme modali anti-simmetriche**, che presentano **diametri nodali**, i cui punti non si spostano durante la vibrazione
- **forme modali miste**, che presentano sia **circonferenze** che **diametri nodali**, i cui punti non si spostano durante la vibrazione





SISTEMI CONTINUI

PIASTRA CIRCOLARE

Le vibrazioni proprie di una piastra circolare di raggio R, sono date dalla seguente relazione:

$$\omega_{n,m} = \frac{\alpha_{n,m}}{R^2}\sqrt{\frac{D}{\rho h}}$$

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

Libera al bordo esterno

α_{nm}	$m=0$	$m=1$	$m=2$
$n=0$	0	0	5,251
$n=1$	9,076	20,52	35,24
$n=2$	38,52	59,86	–

Incastrata al bordo esterno

α_{nm}	$m=0$	$m=1$	$m=2$
$n=0$	10,21	21,22	34,84
$n=1$	39,78	60,84	84,58
$n=2$	88,90	120,08	153,81