A beam-theory based method to partition fracture modes in delaminated beams

Paolo S. Valvo

1Department of Civil Engineering (Structures), University of Pisa, Italy
E-mail: p.valvo@ing.unipi.it

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SUMMARY. The paper presents a beam-theory based method to partition fracture modes in planar laminated beams affected by through-the-width delaminations. According to classical laminated beam theory, the axial, shear and bending deformabilities, as well as bending-extension coupling, are taken into account. The kinematics of crack growth is analysed by defining the crack-tip displacement rates as the relative displacements at the crack tip per unit crack extension. Besides, the crack-tip forces exchanged between the separating laminates are computed. Lastly, by considering the work done by the crack-tip forces for the corresponding crack-tip displacement rates, explicit expressions for the energy release rate and its modal contributions are deduced.

1 INTRODUCTION

Classical laminated plate theory [1] is commonly applied in the analysis of delamination fracture in composites [2, 3], since the delaminated laminates are modelled as assemblages of sublaminates connected by rigid or deformable joints and interfaces. Besides, delamination growth criteria usually assume that crack propagation occurs when the energy release rate, $G$, reaches a critical value, $G_c$ [4]. In general, however, delamination cracks propagate under mixed-mode fracture conditions, so it becomes necessary to partition the energy release rate into two additive contributions, $G_I$ and $G_{II}$, related to fracture modes I (opening) and II (sliding), respectively. To this end, various alternative, but not equivalent, methods have been proposed.

For rigidly connected sublaminates, Williams [5] developed a global method to partition the energy release rate, based on analysis of the global forces acting on the cracked laminate. Schapery and Davidson [6] observed that Williams’ assumptions were not generally fulfilled for asymmetrically delaminated laminates and proposed a method based on classical plate theory. Alternatively, Suo and Hutchinson developed a local method [7], where the mode mixity, i.e. the ratio $G_{II}/G_I$, is determined by analysing the singular stress field at the crack tip of a semi-infinite crack between two infinite isotropic elastic layers. Recently, the local method has been extended to include the effects of shear forces at the crack tip [8] and orthotropic materials [9].

On the other hand, if the sublaminates are connected by a deformable interface, the modal contributions to $G$ can be computed directly, based on the (peak) values of the interfacial stresses at the crack tip [10–13]. Nonetheless, Qiao and Wang [14], yet considering a deformable interface, proposed to evaluate the mode mixity via an adaptation of the local method. This approach appears somehow questionable since the local method, originally developed in the context of plane elasticity, assumes that a stress singularity is present at the crack tip, but this hypothesis does not hold true when the sublaminates are connected by a deformable (elastic) interface.

In this paper we show how the energy release rate associated with the growth of a delamination in a laminated beam can be partitioned into its modal contributions within the context of beam theory. To this aim, we consider a laminated beam, affected by a through-the-width delamination,
as an assemblage of three rigidly connected laminated beams. Each beam is modelled according to classical laminated beam theory, while taking into account the axial, shear and bending deformabilities, as well as bending-extension coupling. Under general load conditions, a small extension of the existing crack is considered. Thus, the kinematics of crack growth is analysed by defining the crack-tip displacement rates as the relative displacements occurring at the crack tip per unit crack extension. Besides, the crack-tip forces exchanged between the separating laminates are computed. Lastly, by considering the work done by the crack-tip forces for the corresponding crack-tip displacement rates, explicit expressions for the energy release rate and its modal contributions are deduced.

One example is presented to illustrate the effectiveness of the method.

2 DELAMINATED BEAM MODELLING

2.1 Beam-theory model of a delaminated laminate

Consider a laminate $AB$ of length $L$, thickness $H = 2h$, and width $B$, affected by a through-the-width delamination of length $a$ (Fig. 1). The delamination runs from the end section $A$ to the intermediate section $C$, to which the crack tip $C$ belongs, thus splitting the laminate into two sublaminates of thicknesses $H_1 = 2h_1$ and $H_2 = 2h_2$, respectively. We denote with $b = L - a$ the length of the unbroken part of the laminate, included between sections $C$ and $B$.

Figure 1: Delaminated laminate subjected to concentrated and distributed loads.

We suppose the laminate to be in equilibrium under the action of a known system of in-plane concentrated and distributed loads. Moreover, we assume that no out-of-plane effects are present, so that the delaminated laminate can be modelled as a planar laminated beam or, more precisely, as an assemblage of three planar laminated beams, each rigidly connected to the others at section $C$ (Fig. 2a). In particular, beams 1 and 2 correspond to the upper and lower sublaminates, respectively, in the delaminated part of the laminate (between sections $A$ and $C$), while beam 3 corresponds to the unbroken part (between sections $C$ and $B$).

A rectangular reference system $Oxz$ is fixed with the origin $O$ at the intersection between the crack-tip cross section and the centreline of the unbroken part of the laminate, the $x$- and $z$-axes aligned with the laminate’s axial and transverse directions, respectively (Fig. 2b). Correspondingly, we indicate with $u_d(x)$ and $w_d(x)$ the axial and transverse displacements of the beams’ centrelines, and with $\phi_d(x)$ their cross sections’ rotations, positive if counter-clockwise (here and in the following the beams are identified by the subscripts $\alpha = 1, 2, 3$).
According to Timoshenko’s beam theory kinematics, we define

\[ \varepsilon_a(x) = \frac{du_a}{dx}, \quad \gamma_a(x) = \frac{dv_a}{dx} + \phi_a(x), \quad \kappa_a(x) = \frac{d\phi_a}{dx}, \quad (1) \]

respectively as the axial strain, shear strain, and curvature. The related internal forces in beams are the axial force, shear force, and bending moment, respectively, given by

\[ N_a(x) = B [ A_a \varepsilon_a(x) + B_a \kappa_a(x) ], \quad Q_a(x) = B C_a \gamma_a(x), \quad M_a(x) = B [ B_a \varepsilon_a(x) + D_a \kappa_a(x) ], \quad (2) \]

where \( A_a, B_a, C_a, \) and \( D_a \) respectively are the extension stiffness, bending-extension coupling stiffness, shear stiffness, and bending stiffness (per unit width) of the beams, computed according to classical laminated plate theory [1]. By inverting Eqs. (2), we obtain also

\[ \varepsilon_a(x) = [ a_a N_a(x) + b_a M_a(x) ] / B, \quad \gamma_a(x) = c_a Q_a(x) / B, \quad \kappa_a(x) = [ d_a N_a(x) + d_a M_a(x) ] / B, \quad (3) \]

where

\[ a_a = \frac{D_a}{A_a D_a - B_a c_a}, \quad b_a = -\frac{B_a}{A_a D_a - B_a c_a}, \quad c_a = \frac{1}{C_a}, \quad d_a = \frac{A_a}{A_a D_a - B_a c_a}, \quad (4) \]

are the extension compliance, bending-extension coupling compliance, shear compliance, and bending compliance of the beams, respectively [1].
2.2 Crack-tip relative displacements and crack-tip displacement rates

Imagine now that a small segment \( S \) of the laminate is cut out in the neighbourhood of the crack tip \( C \) (Fig. 3a). Regardless of the actual load system applied to the laminate, if we exclude the presence of concentrated loads at the crack-tip cross section, the segment \( S \) will be in equilibrium under the internal forces acting on the cross sections close to the crack tip. Thus, for \( x \to 0 \), if we denote with \( N_1, Q_1, M_1 \) and \( N_2, Q_2, M_2 \) the internal forces in beams 1 and 2, respectively, the internal forces in beam 3 will be

\[
N_3 = N_1 + N_2, \quad Q_3 = Q_1 + Q_2, \quad M_3 = M_1 + M_2 - N_1h_2 + N_2h_1.
\]

Next, suppose that the crack propagates in a self-similar way, increasing its length by a small amount, \( \Delta a \). Hence, the crack-tip segment \( S \) transforms into the segment \( S_0 \) (Fig. 3b), where the crack tip reaches a new position, identified by point \( D \), and the point \( C \) splits into two points, \( C_1 \) and \( C_2 \), belonging to beams 1 and 2, respectively. If crack growth occurs under fixed load conditions, the internal forces in the cross sections close to the crack tip do not change appreciably and \( S_0 \) can still be considered in equilibrium under the same internal forces acting on \( S \).

\[
\Delta u = u_{c2} - u_{c1} = (u_i - \phi h_i) - (u_i + \phi h_i),
\]

\[
\Delta w = w_{c2} - w_{c1} = w_2 - w_1,
\]

\[
\Delta \phi = \phi_{c2} - \phi_{c1} = \phi_2 - \phi_1,
\]

where all the (generalised) displacements are tacitly evaluated at the crack-tip section \( x = 0 \).
By solving the auxiliary problem of a laminated cantilever beam loaded at its end (details are here omitted for brevity), it is easily shown that the crack-tip relative displacements are

\[ \Delta u \equiv \frac{1}{B} \left[ (a_1 + b_1 h_1) N_1 - (a_2 - b_2 h_2) N_2 + (b_1 + d_1 h_1) M_1 - (b_2 - d_2 h_2) M_2 \right] \Delta a, \]

\[ \Delta w \equiv \frac{1}{B} (c_1 Q_1 - c_2 Q_2) \Delta a, \]

\[ \Delta \phi \equiv \frac{1}{B} (b_2 N_1 - b_2 N_2 + d_1 M_1 - d_2 M_2) \Delta a, \]

where higher-order powers of \( \Delta a \) have been neglected.

In order to eliminate the dependence on \( \Delta a \), we define the crack-tip displacement rates as

\[ \eta_u = \lim_{\Delta a \to 0} \frac{\Delta u}{\Delta a} \equiv \frac{1}{B} \left[ (a_1 + b_1 h_1) N_1 - (a_2 - b_2 h_2) N_2 + (b_1 + d_1 h_1) M_1 - (b_2 - d_2 h_2) M_2 \right], \]

\[ \eta_w = \lim_{\Delta a \to 0} \frac{\Delta w}{\Delta a} = \frac{1}{B} (c_1 Q_1 - c_2 Q_2), \]

\[ \eta_\phi = \lim_{\Delta a \to 0} \frac{\Delta \phi}{\Delta a} = \frac{1}{B} (b_2 N_1 - b_2 N_2 + d_1 M_1 - d_2 M_2). \]

By recalling Eqs. (2) and (4), it can be shown that

\[ \eta_u = \epsilon_1 - \epsilon_2 + \kappa_1 h_1 + \kappa_2 h_2, \quad \eta_w = \gamma_1 - \gamma_2, \quad \eta_\phi = \kappa_1 - \kappa_2, \]

where all the strain measures are evaluated at the crack-tip section \( x = 0 \).

The crack-tip displacement rates appear as very helpful tools for fracture mechanics problems, since they sum up a description of the kinematics of crack growth and, in particular, will play a crucial role in the partition of the energy release rate into its modal contributions.
2.3 Crack-tip forces

The displacement compatibility of system $S$ can be recovered from $S_0$ by superimposing to the latter an auxiliary system, $S_C$, where suitable axial forces, $N_C$, transverse forces, $Q_C$, and couples, $M_C$, are exchanged between points $C_1$ and $C_2$ (Fig. 5).

![Figure 5: Crack-tip forces in system $S_C$.](image)

The intensities of the above crack-tip forces are determined in such a way as to restore displacement compatibility previous to crack growth. To this aim, we compute the crack-tip displacement rates associated with system $S_C$, which turn out to be

$$
\begin{align*}
\eta^C_x &= -N_C \eta^X_x - M_C \eta^M_x, \\
\eta^C_y &= -Q_C \eta^D_y, \\
\eta^C_\phi &= -N_C \eta^N_\phi - M_C \eta^M_\phi,
\end{align*}
$$

where

$$
\begin{align*}
\eta^N_x &= \frac{1}{B} (a_1 + a_2 + 2b_1 h_1 - 2b_2 h_2 + d_1 h_1^2 + d_2 h_2^2), \\
\eta^M_x &= \eta^N_x = \frac{1}{B} (b_1 + b_2 + d_1 h_1 - d_2 h_2), \\
\eta^N_y &= \frac{1}{B} (d_1 + d_2), \\
\eta^M_y &= \frac{1}{B} (c_1 + c_2),
\end{align*}
$$

are generalised compliances, which describe the deformability of the crack-tip element.

Then, we require that

$$
\eta_x + \eta^C_x = 0, \quad \eta_y + \eta^C_y = 0, \quad \eta_\phi + \eta^C_\phi = 0.
$$

By substituting Eqs. (10) into (12) and solving for the crack-tip forces, we obtain

$$
\begin{align*}
N_C &= \frac{\eta^N_x \eta_\phi - \eta^M_x \eta_y}{\eta^N_x \eta_\phi - \eta^M_x \eta_y}, \\
Q_C &= \frac{\eta^N_y \eta_\phi - \eta^M_y \eta_y}{\eta^N_x \eta_\phi - \eta^M_x \eta_y}, \\
M_C &= \frac{\eta^N_\phi \eta_y - \eta^M_\phi \eta_y}{\eta^N_x \eta_\phi - \eta^M_x \eta_y},
\end{align*}
$$

(13)
3 ENERGY RELEASE RATE AND FRACTURE MODE PARTITIONING

3.1 Energy release rate

Under fixed displacements [2], the energy release rate associated with crack growth is

\[ G = -\frac{1}{B} \lim_{\Delta a \to 0} \frac{\Delta U}{\Delta a}, \]  

(14)

where \( \Delta U \) is the change in strain energy related to the increase in crack length \( \Delta a \). According to the definitions given in the previous section,

\[ \Delta U = U - U_0, \]  

(15)

where \( U \) and \( U_0 \) are the strain energies in systems \( S \) and \( S_0 \), respectively. Since system \( S \) can be obtained by superimposing systems \( S_0 \) and \( S_C \), it is also

\[ U = U_0 + U_C, \]  

(16)

where \( U_C \) is the strain energy in system \( S_C \). By substituting Eq. (15) and (16) into (14), we obtain

\[ G = \frac{1}{B} \lim_{\Delta a \to 0} \frac{U_C}{\Delta a}. \]  

(17)

The strain energy stored in \( S_C \) can be evaluated by direct calculation, however it is more convenient to apply Clapeyron’s theorem, which yields

\[ U_C = \frac{1}{2} (N_c \Delta u + Q_c \Delta w + M_c \Delta \phi), \]  

(18)

where \( \Delta u, \Delta w, \) and \( \Delta \phi \) are the crack-tip relative displacements in system \( S_0 \), given by Eqs. (7), which are equal in magnitude and opposite in sign to those caused by the crack-tip forces in \( S_C \).

By substituting Eq. (18) into (17), and remembering Eqs. (8), we obtain the energy release rate as a function of the crack-tip forces and crack-tip displacement rates,

\[ G = \frac{1}{2B} (N_c \eta_u + Q_c \eta_w + M_c \eta_\phi). \]  

(19)

Furthermore, by substituting Eqs. (12) into (19), the energy release rate can be expressed in terms of the crack-tip forces only,

\[ G = \frac{1}{2B} [\eta_u N_c \eta_u + (\eta_w + \eta_\phi) N_c M_c + \eta_\phi N_c \eta_\phi + \eta_u M_c \eta_u], \]  

(20)

or, by substituting Eqs. (13) into (19), in terms of the crack-tip displacement rates only,
3.2 Fracture mode partitioning

In planar fracture mechanics problems the energy release rate can be decomposed as

$$ G = G_I + G_{II}, \quad (22) $$

where the addends $G_I$ and $G_{II}$ are related to the so-called opening and sliding fracture modes, respectively. In particular, the *opening mode*, or *mode I*, corresponds to a fracture process where the separating parts of material move away one from another perpendicularly to the direction of crack propagation; while the *sliding mode*, or *mode II*, occurs when the separating parts undergo a relative displacement parallel to the direction of crack propagation.

In our model, mode I is related to the crack-tip displacement rates $\eta_w$ and $\eta_\phi$, while mode II is related to $\eta_u$. However, some caution is required in order to correctly recognise the contributions to $G$ related to each fracture mode. If we closely examine both Eqs. (20) and (21), it is apparent that the terms depending on $Q_C$ and $\eta_w$ contribute to $G_I$ only (incidentally, these terms are relevant only if we include shear deformability in the analysis); on the other hand, the terms depending on $N_C$, $M_C$ and $\eta_u$, $\eta_\phi$ are strongly tied one another and, hence, contribute to both fracture modes.

The key to solve the enigma of fracture mode partitioning is to start from determining the contribution $G_{II}$, which must be in the following form:

$$ G_{II} = \frac{1}{2B} N_{C}^{II} \eta_u, \quad (23) $$

where

$$ N_{C}^{II} = \frac{\eta_u}{\eta_\phi}, \quad (24) $$

is the crack-tip axial force that would be able to cancel the crack-tip sliding displacement rate, $\eta_\phi$, if no crack-tip couple, $M_C$, were present. Remembering Eqs. (11) and (12), we observe that $N_{C}^{II}$ is in general distinct from $N_C$, since the latter also contributes to cancel $\eta_\phi$ but they coincide if $\eta_\phi^w = \eta_\phi^0 = 0$. This happens, for instance, when the delaminated sublaminates are uncoupled in bending-extension ($b_1 = b_2 = 0$) and such that $d_1 h_1 = d_2 h_2$.

By substituting Eq. (24) into (23) and (23) into (22), we finally obtain the explicit expressions of the modal contributions to the energy release rate in terms of the crack-tip displacement rates

$$ G_I = \frac{1}{2B} \left( \frac{(\eta_\phi - \eta_\phi^0)^2 \eta_u^2}{\eta_\phi^0 - \eta_\phi^w \eta_u^0 \eta_\phi^w - \eta_\phi^w} + \frac{\eta_u^2}{\eta_\phi^0} \right), \quad G_{II} = \frac{1}{2B} \frac{\eta_u^2}{\eta_\phi^0}, \quad (25) $$

Expressions of $G_I$ and $G_{II}$ as functions of the crack-tip forces are here omitted for brevity.
4 AN EXAMPLE OF APPLICATION: THE ADCB TEST

As a first example of application, we consider the asymmetric double cantilever beam (ADCB) test (Fig. 6), used to measure the mixed-mode fracture toughness of composite laminates [15]. According to the global method [5], the ADCB test should be a case of pure mode I, however this prediction is contradicted by all other methods of analysis, such as the local method, the virtual crack closure technique, the elastic-interface based models and so on, which correctly distinguish the mixed-mode character of this test (see Ref. [13] for a detailed discussion on this topic).

Figure 6: The asymmetric double cantilever beam (ADCB) test.

The internal forces at the crack-tip sections in this case are

\[ N_1 = 0, \quad Q_1 = P, \quad M_1 = Pa; \quad N_2 = 0, \quad Q_2 = -P, \quad M_2 = -Pa. \] (26)

By assuming that the delaminated laminates are uncoupled \((b_1 = b_2 = 0)\), from Eqs. (8) we compute the crack-tip displacement rates,

\[ \eta_\theta = \frac{Pa}{B}(d_1h_1 - d_1h_2), \quad \eta_\phi = \frac{Pa}{B}(c_1 + c_2), \quad \eta_\psi = \frac{Pa}{B}(d_1 + d_2). \] (27)

and from Eqs. (11) we get the crack-tip compliances,

\[ \eta_\omega = \frac{1}{B}(a_1 + a_2 + d_1h_1^2 + d_1h_2^2), \quad \eta_\omega' = \frac{1}{B}(d_1h_1 - d_1h_2), \quad \eta_\omega'' = \frac{1}{B}(d_1 + d_2), \]
\[ \eta_\omega'' = \frac{1}{B}(c_1 + c_2). \] (28)

Hence, Eqs. (25) yield the mode I and II contributions to the energy release rate

\[ G_I = \frac{P^2a^2}{2B^2}(d_1 + d_2) - \frac{(d_1h_1 - d_1h_2)^2}{a_1 + a_2 + d_1h_1^2 + d_1h_2^2} \] + \[ \frac{P^2}{2B^2}(c_1 + c_2), \quad G_{II} = \frac{P^2a^2}{2B^2}(d_1 + d_2), \] \[ \eta_\psi = \frac{1}{B}(c_1 + c_2). \] (29)

To confirm that the above result is correct, we notice that Eqs. (29) are identical with those that can be obtained from the elastic-interface model of the ADCB test [13] in the limit case of a rigid interface (i.e. when the elastic constants of the interface go to infinity).
5 CONCLUSIONS

We have presented a beam-theory based method to partition fracture modes in planar laminated beams affected by through-the-width delaminations. According to classical laminated beam theory, the axial, shear and bending deformabilities have been taken into account. Moreover, our analysis has included also bending-extension coupling, which, to our knowledge, has never been considered before by the fracture mode partitioning methods proposed in literature.

The kinematics of crack growth has been analysed by defining the crack-tip displacement rates as the relative displacements occurring at the crack tip per unit crack extension. These quantities appear as very helpful tools for fracture mechanics problems, since they sum up a description of the kinematics of crack growth. In this paper, we have used the crack-tip displacement rates to determine the explicit expressions of the crack-tip forces, energy release rate and modal contributions. Extension to planar and spatial elasticity problems looks promising.

Due to length restrictions, only one applicative example could be presented here. A more detailed explanation of the method and more examples will be presented in an extended paper.

References