Note del corso "Aspetti matematici della fluidodinamica"

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Chapter 1

The physics of the Navier-Stokes equations and a few heuristic computations

1.1 Introduction

This series of lectures is devoted to the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \triangle u + f \tag{1.1}$$
$$\mathtt{div} u = 0.$$

These equations are usually accepted as a model for incompressible, viscous, constant-density fluids, like water or air at normal conditions. Assume the fluid occupies a region D of $\mathbb{R}^3$. Then

- $u = (u_1, u_2, u_3) : D \times [0, \infty) \to \mathbb{R}^3$ is the velocity field of the fluid ($u(x,t)$ is the velocity at point x and time t),
- $p : D \times [0, \infty) \to \mathbb{R}$ is the pressure field, and both are considered as the unknown of the equation,
- $f : D \times [0, \infty) \to \mathbb{R}^3$ is the force acting on the body of the fluid,
- the constant $\nu > 0$ is called kinematic viscosity.

About notations, recall that $\mathtt{div} u = \sum_{k=1}^{d} \frac{\partial u_k}{\partial x_k}$; we write $u \cdot \nabla u$ for the vector field $\sum_{k=1}^{d} u_k \frac{\partial u}{\partial x_k}$; the Laplacian $\triangle$ is the differential operator $\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ and $\triangle u$ is understood componentwise. In other words, the i-component of the Navier-Stokes equations, $i = 1, 2, 3$, is

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{d} u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = \nu \sum_{k=1}^{d} \frac{\partial^2 u_i}{\partial x_k^2} + f_i.$$

Also, we shall write ∂_k or D_k for $\frac{\partial}{\partial x_k}$. However, we prefer to work with vector notations.

1.1.1 Meaning of the equations

In a sentence, the equation $\mathtt{div} u = 0$ states that the fluid is not compressible and the evolution system for u and p is essentially Newton second law (along particle trajectories). Although it is not our aim to explain in detail the physics of fluids and to this end we address the reader to books like [6], [10], let us add a few more words about the meaning of the terms in the equations.

Originally one should also introduce a variable $\rho : D \times [0, \infty) \to \mathbb{R}$ representing mass density, that in the model above is assumed to be constant and normalized to one. Call *fluid particle* a very small volume of fluid, ideally pointwise (notice: a fluid particle is not an atom or molecule, but something like a set of 10^{18} molecules, still extremely small from the macroscopic viewpoint). Consider the motion of a fluid particle. If X_t^x is the position at time t of the particle that started at time zero from position x, then

$$\frac{dX_t^x}{dt} = u\left(X_t^x, t\right).$$

Newton second law for the fluid particle is

$$\rho\left(X_t^x, t\right) \frac{d^2 X_t^x}{dt^2} = F\left(X_t^x, t\right)$$

where $F\left(X_t^x, t\right)$ is the force acting on the particle at time t. The second derivative $\frac{d^2 X_t^x}{dt^2}$ is equal to $\frac{du(X_t^x, t)}{dt}$ namely to the so called *material derivative* of u

$$\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right)\left(X_t^x, t\right).$$

It is the acceleration of the particle. Thus we have

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)=F$$

computed either along the fluid particle trajectory, or equivalently at any space-time point (x,t).

Three forces act on fluid elements: pressure, friction and body forces (like gravitational force).

We do not define the concept of pressure here but just recall that at any point of a fluid a scalar quantity p called pressure is defined; it produces a normal force of intensity p orthogonal to any ideal or real plane passing through that point. If in a region the pressure is constant, no net pressure force is applied to any fluid element, since pressure force acts in all directions in the same way. On the contrary, if there is a gradient of pressure $\nabla p\,(x)$ at point x, a fluid element at position x is subject to a pressure force because of the (infinitesimal) difference in pressure on different sides of the fluid element. This is an explanation in plain words of the fact that pressure acts through its gradient.

Including the body force denoted by $f(x,t)$, we have reached the equations

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)+\nabla p=F_{friction}+f$$

where $F_{friction}$ denotes the friction force. If no friction exists we call the fluid *ideal* and the previous equations are called the Euler equations.

Exercise 1 *Consider a one-dimensional fluid, namely a fluid in $\mathbb{R}^3$ such that all fields depend only on the first coordinate x_1 and $u_2=u_3=0$. Also $\nabla p=(\partial_1 p,0,0)$. Check in this simple case that the reasonable convention is to write ∇p on the right-hand-side of the equations, as we have done above.*

The full story around the mathematical expression of friction forces is too long for this introduction, so we content ourselves with an analogy with other theories that the reader could know. Any time, in a partial differential equation, we have the Laplacian $\triangle u$ of a variable u, the effect is that of a *diffusion* of that quantity. Intuitively this is what happens to the velocity field of the fluid too, due to friction: fluid particles moving faster but being in contact with fluid particles moving slower, give them part of their momentum (thus velocity diffuses). In fact at the microscopic level this is more or

less what happens. Using rational mechanics it is possible to justify in more firm way that $F_{friction}$ is given by $\triangle u$, multiplied by a constant (viscosity coefficient) typical of the particular fluid of interest. We have reached the Navier-Stokes equations of the previous section. As we have already said, we consider fluids at regimes where the density remains essentially constant, so we assume ρ constant, and we set it equal to one to simplify the mathematical expressions. Setting $\rho = 1$ amounts to introduce the so called reduced pressure $p = \frac{p'}{\rho}$ where p' was the original pressure discussed above. And also the viscosity coefficient becomes divided by ρ, giving rise to the kinematic viscosity ν.

The equation $\mathtt{div} u = 0$ describes the incompressibility of the fluid. If A is a small region, we have

$$\int_A \mathtt{div} u dx = \int_{\partial A} u \cdot n d\sigma$$

where n is the outer normal to the boundary. The integral $\int_{\partial A} u \cdot n d\sigma$ is the total flow of fluid through the boundary of A. Hence $\mathtt{div} u = 0$ means that the total flow through any closed boundary is zero, a way to understand incompressibility.

One could also start from the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathtt{div}\,(\rho u) = 0$$

which expresses the conservation of mass. When ρ is assumed constant, we immediately get $\mathtt{div} u = 0$.

1.1.2 Boundary conditions

Realistic examples of fluids usually have complicate boundary conditions, like tubes and rivers, jets, volcanos, etc., with inflow and outflow of fluid, or maybe D is an external domain (the exterior of an airplane) with constant non zero values at infinity. In order to develop a mathematical theory it is more reasonable to start with less wild cases, with the hope that part of the theory extends to real examples with suitable modifications. We shall treat three cases. The simplest one, non-realistic but useful to reach advanced results very fast, is the case $D = d$-dimensional torus. In other words, we may think that the fluid occupies the full space $\mathbb{R}^d$ but all fields f, u and p

are L-periodic, namely

$$f(x + kL, t) = f(x, t)$$

for all $k \in \mathbb{Z}^d$, and similarly for u and p. The torus has two technical advantages: it is compact, it has no boundary. Remarkable is that theoretical physicists interested in turbulence consider very often fluids on tori; we shall come back to this point.

Partially easy like the torus and partially unrealistic as well is the case $D = \mathbb{R}^d$. It shares the advantage of no boundary, but lacks compactness. Apparently it looks much more realistic but a lot depends on the conditions at infinity: if we assume that fields decay to zero at infinity, then we have the advantages of no boundary, but we loose most interesting physical examples. If on the contrary we accept non-zero values at infinity, even increasing and fluctuating, then physically we go in a very interesting direction, but mathematically it is more difficult than many other cases. Except when explicitly stated, if we deal with $D = \mathbb{R}^d$ we mean the simple case of fields decay to zero at infinity.

Finally, the classical, say canonical, case in the mathematical literature is D equal to a bounded open domain with some degree of regularity of the boundary (Lipschitz is sufficient for most of the results). Viscous fluids moving near solid boundaries are observed to be at rest exactly on the boundary, although this may appear strange at first sight. Looking carefully, there is dust on surfaces even with strong wind. The velocity profile of water slowly going through a conduct is parabolic, not constant. The physically natural condition is thus the so called *non-slip* boundary condition, namely

$$u = 0 \text{ on } \partial D.$$

Such a condition introduces several mathematical difficulties which reflect basic novelties from the physical viewpoint. Think to a quite fast fluid, and notice that u passes from the value $u = 0$ on ∂D to large values just quite close to ∂D. Thus u has large gradients near the boundary. Intuitively this may be the origin of unstable behaviors, turbulence, creation of vortices. Now we understand why the torus or the full space is easier. The reason why also theoretical physicists analyze tori is that one hope to simulate the effect of a boundary by means of suitable forces f. And also that certain phenomena, like isotropic turbulence, cannot occur in asymmetric regions like near a boundary.

Let us mention the usual boundary conditions for the Euler equations (non viscous fluids, $\nu = 0$) are the so called *slip* conditions, namely $u \cdot n = 0$.

1.2 Global energy balance

In this and the following sections we perform heuristic computations on solutions to the 3D Navier-Stokes equations in order to guess a number of results. Some of the estimates we obtain here will correspond to the best known rigorous results. Other facts like the energy identity, are open problems at the rigorous level.

From now on we start to use several natural notations. We denote by $|u(x,t)|$ the norm of the 3D vector $u(x,t)$, and by $\|\nabla u(x,t)\|$ the following norm of the matrix $\nabla u(x,t)$:

$$\|\nabla u(x,t)\|^2 = \sum_{i,j=1}^{3} |\partial_i u_j|^2 = \sum_{i=1}^{3} |\nabla u_j|^2 .$$

Sometimes we shall also use the notation $\|x\|$ for the Euclidean norm of an element $x \in \mathbb{R}^3$ if we feel the risk that it is confused with the absolute value of a real number.

Let (u,p) be a sufficiently regular solution to the 3D Navier-Stokes equations in a domain D; a priori here D can be any measurable set of $\mathbb{R}^3$. We multiply the equations by u, integrate on a regular set $B \subset D$ and deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_B |u|^2 dx + \int_B (u\cdot\nabla u)\cdot u dx + \int_B u\cdot\nabla p dx \\ &= \nu\int_B \triangle u\cdot u dx + \int_B u\cdot f dx. \end{aligned}$$

We can also write $\int_B (u\cdot\nabla u)\cdot u dx$ in the form

$$\frac{1}{2}\int_B \left(u\cdot\nabla |u|^2\right) dx.$$

Now we integrate by parts in the second, third and forth integral and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_B |u|^2\,dx + \frac{1}{2}\int_{\partial B} |u|^2\, u\cdot n d\sigma - \frac{1}{2}\int_B |u|^2\,\mathtt{div} u dx\\
&+\int_{\partial B} pu\cdot n d\sigma - \int_B p\mathtt{div} u dx\\
&= -\nu\int_B \|\nabla u\|^2\,dx + \nu\int_{\partial B}\frac{\partial u}{\partial n}\cdot u dx + \int_B u\cdot f dx.
\end{aligned}$$

where n is the outer normal. In particular we have used the identity

$$\int_B \triangle u_i u_i dx = \int_{\partial B}\frac{\partial u_i}{\partial n}u_i dx - \int_B |\nabla u_i|^2\,dx.$$

Thanks to $\mathtt{div} u = 0$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_B |u|^2\,dx + \nu\int_B \|\nabla u\|^2\,dx = \int_B u\cdot f dx\\
&-\frac{1}{2}\int_{\partial B}|u|^2\, u\cdot n d\sigma - \int_{\partial B} pu\cdot n d\sigma + \nu\int_{\partial B}\frac{\partial u}{\partial n}\cdot u dx.
\end{aligned}$$

This identity greatly simplifies if we take $B = D$ and we assume that u satisfies proper boundary conditions. Both in the case of a bounded domain with non-slip boundary condition, in the case of $D = [0, L]^3$ with periodic boundary conditions, and in the case $D = \mathbb{R}^3$ with suitable decay of u at infinity, the boundary terms are equal to zero and we get the *energy identity*

$$\frac{1}{2}\frac{d}{dt}\int_D |u|^2\,dx + \nu\int_D \|\nabla u\|^2\,dx = \int_D u\cdot f dx.$$

In integral form in time:

$$\begin{aligned}
&\frac{1}{2}\int_D |u(x,t)|^2\,dx + \nu\int_0^t\int_D \|\nabla u\|^2\,dxds\\
&= \frac{1}{2}\int_D |u(x,0)|^2\,dx + \int_0^t\int_D u\cdot f dxds.
\end{aligned} \tag{1.2}$$

It states that kinetic energy is not conserved, unless $\nu = 0$ and $f = 0$. The variation of kinetic energy is equal to the sum of the energy dissipated by friction and the work done by the force f. We advise that this identity

will not be proved rigorously in the sequel (it is an open problem). On the contrary, the energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_D |u|^2\,dx + \nu\int_D \|\nabla u\|^2\,dx \le \int_D u\cdot f dx$$

is something reachable, in the sense that we can prove the existence of solutions satisfying it.

Moreover, certain consequences of the energy identity or inequality will also correspond to theorems. Let us mention a very important one, absolutely basic to address the definition of weak solution given in the next section. For simplicity, let us work under the assumption

$$K_0 := \frac{1}{2}\int_D |u(x,0)|^2\,dx < \infty$$
$$K_1 := \frac{1}{2}\int_0^T\int_D |f(x,t)|^2\,dxdt < \infty$$

although the result is true for more general f. We easily have

$$2\int_0^t\int_D u\cdot f dxds \le \int_0^t\int_D |u|^2\,dxds + 2K_1$$

hence from the energy inequality

$$\frac{1}{2}\int_D |u(x,t)|^2\,dx \le K_0 + K_1 + \frac{1}{2}\int_0^t\int_D |u(x,s)|^2\,dxds.$$

We apply Gronwall lemma (see the appendix) to $\frac{1}{2}\int_D |u(x,t)|^2\,dx$ and get

$$\frac{1}{2}\int_D |u(x,t)|^2\,dx \le (K_0+K_1)\,e^t.$$

Moreover, again from the energy inequality we have

$$\nu\int_0^T\int_D \|\nabla u\|^2\,dxdt \le K_0 + \int_0^T\int_D u\cdot f dxds$$

and thus

$$\nu\int_0^T\int_D \|\nabla u\|^2\,dxdt \le K_0 + K_1 + \frac{1}{2}\int_0^t\int_D |u(x,s)|^2\,dxds$$

where now we may estimate $\int_D |u(x,s)|^2\,dx$ by the previous upper bound. Putting all together, we get

$$\sup_{t\in[0,T]}\int_D |u(x,t)|^2\,dx \le 2\left(K_0+K_1\right)e^T$$
$$\int_0^T\int_D \|\nabla u\|^2\,dxdt \le \frac{1}{\nu}\left(K_0+K_1\right)\left(1+Te^T\right).$$

These are not even the best possible estimates we may get, but they suggest that, under the assumptions $K_0<\infty$, $K_1<\infty$, we could find a solution u with

$$\sup_{t\in[0,T]}\int_D |u(x,t)|^2\,dx + \int_0^T\int_D \|\nabla u\|^2\,dxdt < \infty. \tag{1.3}$$

In the next chapter we shall give a definition of weak solution of the Navier-Stokes equations that is essentially based on this regularity property. We shall also prove that such weak solutions exist. Up to date, there is no proof of existence of global solutions with more regularity (except for side properties that we shall discuss at due time). Thus the level of regularity described by (1.3) must be considered as the basic starting point of all our investigations.

Property (1.3) states that the kinetic energy is bounded at any time, and uniformly in time. The integral $\int_D \|\nabla u(x,t)\|^2\,dx$ does not necessarily have the same feature: it may blow-up in time, in a integrable way. To make an example (artificial, not solution to (1.1)), consider the function

$$u(x,t)=\alpha(t)^{3/2}\varphi(x\alpha(t)),\quad \varphi,\nabla\varphi\in L^2\left(\mathbb{R}^3\right),\alpha\in L^2(0,T).$$

We have $\nabla u(x,t)=\alpha(t)^{5/2}(\nabla\varphi)(x\alpha(t))$,

$$\int_{\mathbb{R}^3}|u(x,t)|^2\,dx=\int_{\mathbb{R}^3}|\varphi(x\alpha(t))|^2\alpha^3(t)\,dx=\int_{\mathbb{R}^3}|\varphi(y)|^2\,dy$$

$$\begin{aligned}\int_{\mathbb{R}^3}\|\nabla u(x,t)\|^2\,dx &= \alpha^2(t)\int_{\mathbb{R}^3}|(\nabla\varphi)(x\alpha(t))|^2\alpha^3(t)\,dx\\ &= \alpha^2(t)\int_{\mathbb{R}^3}\|\nabla\varphi\|^2\,dx.\end{aligned}$$

Thus (1.3) holds. Depending on α, we see that $\int_{\mathbb{R}^3}\|\nabla u(x,t)\|^2\,dx$ may be unbounded in t even if $\int_{\mathbb{R}^3}|u(x,t)|^2\,dx$ is bounded.

In classical courses of calculus we are used to rather regular functions, so one could ask whether it is easy to give examples of functions $u(x,t)$ having the weak regularity (1.3) and not much more. The previous simple example shifts the problem to the question whether it is easy to give examples of functions $\alpha \in L^2(0,T)$ and $\varphi \in L^2(\mathbb{R}^3)$ with $\nabla\varphi \in L^2(\mathbb{R}^3)$ which are not much more regular. The usual way to build examples is by means of isolated singularities. Thus

$$\alpha(t) = |t-t_0|^{-\beta}, \quad \beta < \frac{1}{2}$$
$$\varphi(x) = e^{-\|x\|}\|x-x_0\|^{-\gamma}, \quad \gamma < \frac{1}{2}$$

are examples (this φ is scalar but one can do similar examples of divergence free vector fields, for instance imposing some symmetries). Could these isolated singularities have something to do with single huge vortices? In the case of a turbulent fluid, we could even image that it develops high values of the velocity field around very many points, not so isolated anymore. We do not know if singularities may really exist in the model of a 3D viscous fluid given by the Navier-Stokes equations.

The existence or absence of singularities is an outstanding open problem (see the presentation of [3] as one of the millennium prize problems). A parallel outstanding open question is the uniqueness of the weak solutions, namely of solutions having essentially only the property (1.3) (the only solutions that have been proved to exist globally in time). Let us show here, heuristically, that easy estimates on the difference of two weak solutions fail to prove uniqueness. Let $\left(u^{(i)}, p^{(i)}\right)$, $i = 1, 2$, be two solutions and let $v = u^{(1)} - u^{(2)}$, $q = p^{(1)} - p^{(2)}$. From (1.1) we have

$$\frac{\partial v}{\partial t} + u^{(1)} \cdot \nabla v + v \cdot \nabla u^{(2)} + \nabla q = \nu \triangle v.$$

The same energy-type computations done above in this section yield (notice in particular that $\int_D v \cdot \left(u^{(1)} \cdot \nabla v\right) dx = 0$ but $\int_D v \cdot \left(v \cdot \nabla u^{(2)}\right) dx$ does not vanish)

$$\frac{1}{2}\int_D |v(x,t)|^2 dx + \nu \int_0^t \int_D \|\nabla v\|^2 dxds$$
$$\leq \frac{1}{2}\int_D |v(x,0)|^2 dx + \left|\int_0^t \int_D v \cdot \left(v \cdot \nabla u^{(2)}\right) dxds\right|.$$

If we could prove an inequality of the form

$$\left|\int_0^t \int_D v \cdot \left(v \cdot \nabla u^{(2)}\right) dxds\right| \tag{1.4}$$
$$\overset{?}{\leq} \frac{\nu}{2} \int_0^t \int_D \|\nabla v\|^2 dxds + \int_0^t \theta^{(2)}(s) \int_D |v(x,s)|^2 dxds$$

where $\theta^{(2)}$ depends on $u^{(2)}$ and if we had $\theta^{(2)} \in L^1(0,T)$ thanks to (1.3) then we would have

$$\frac{1}{2}\int_D |v(x,t)|^2 dx \leq \frac{1}{2}\int_D |v(x,0)|^2 dx + \int_0^t \theta^{(2)}(s) \int_D |v(x,s)|^2 dxds.$$

Gronwall lemma would imply

$$\frac{1}{2}\int_D |v(x,t)|^2 dx \leq \frac{1}{2}\int_D |v(x,0)|^2 dx \cdot e^{\int_0^t \theta^{(2)}(s)ds}.$$

If $v(x,0) = 0$, this implies $v(x,t) = 0$ for every t, namely uniqueness.

But this is not possible. There are many ways to handle the trilinear term but all fail to produce a bound of the form (1.4). Instead of showing some of them, we give a sort of heuristic counterexample: property (1.3) does not imply

$$\int_0^T \int_D \left(|u|^2 |\nabla u|\right) dxdt < \infty. \tag{1.5}$$

Standing this fact, it looks hopeless to estimate $\left|\int_0^t \int_D v \cdot \left(v \cdot \nabla u^{(2)}\right) dxds\right|$ in terms of (1.3) and similar topologies for v.

To see that (1.3) does not imply (1.5), let us use again the example above:

$$\int_0^T \int_{\mathbb{R}^3} \left(|u|^2 |\nabla u|\right) dxdt = \int_0^T \alpha(t)^{5/2} \int_{\mathbb{R}^3} \left(|\varphi(x\alpha(t))|^2 |\nabla\varphi(x\alpha(t))|\right) \alpha(t)^3 dxdt$$
$$= \int_0^T \alpha(t)^{5/2} dt \int_{\mathbb{R}^3} \left(|\varphi(x)|^2 |\nabla\varphi(x)|\right) dx.$$

Now, it is sufficient that $\alpha \in L^2(0,T)$ is not of class $L^{5/2}(0,T)$ and this integral is infinite.

Exercise 2 *Consider the function*

$$u(x,t) = \alpha(t)\,\varphi(x\alpha(t)), \quad \nabla\varphi \in L^2\left(\mathbb{R}^3\right), \alpha \in L^1(0,T).$$

Prove that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u(x,t)|^3\,dx + \int_0^T \int_{\mathbb{R}^3} \|\nabla u\|^2\,dxdt < \infty \tag{1.6}$$

and also

$$\int_0^T \int_{\mathbb{R}^3} \left(|u|^2\,|\nabla u|\right) dxdt < \infty.$$

Is the gap between (1.3) and (1.4) so large? The exercise hints that (1.6) could be sufficient. This is the case! Uniqueness can be proved under this condition, not far from (1.3). However, the distance between the two conditions is finite, although small.

1.3 Local energy balance

The first form of local energy balance has been obtained in the previous section: for any regular bounded open set B we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_B |u|^2\,dx + \frac{1}{2}\int_{\partial B} |u|^2\,u\cdot n d\sigma + \nu\int_B \|\nabla u\|^2\,dx \\ &= \int_B u\cdot f dx - \int_{\partial B} pu\cdot n d\sigma + \nu\int_{\partial B}\frac{\partial u}{\partial n}\cdot u d\sigma. \end{aligned} \tag{1.7}$$

We can also write

$$\nu\int_{\partial B}\frac{\partial u}{\partial n}\cdot u d\sigma = \frac{\nu}{2}\int_{\partial B}\frac{\partial |u|^2}{\partial \nu} d\sigma.$$

The local energy identity states that the variation of kinetic energy in a fixed domain B plus the flow of kinetic energy through the boundary, is balanced by work done by the pressure on the boundary, plus dissipation in B, plus work done by f, plus a boundary term of work associated to frictional forces.

We can also write down a pointwise energy identity, which is a partial differential equation satisfied by the variable

$$E(x,t) = \frac{1}{2}|u(x,t)|^2.$$

One way is to inspect the previous computations, but let us do them again in a slightly different way. Let us formally compute the variation of kinetic energy along trajectories:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)E\\&=u\cdot\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)\\&=u\cdot\left(-\nabla p+\triangle u+f\right).\end{aligned}$$

This is the work done by forces on fluid particle. We can use the identity

$$u\cdot\triangle u=\frac{1}{2}\triangle\left|u\right|^2-\left\|\nabla u\right\|^2$$

to write

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla-\triangle\right)E+\left\|\nabla u\right\|^2=u\cdot\left(f-\nabla p\right). \tag{1.8}$$

This is the partial differential equation for the energy. We could use it as the starting point of many computations of these sections.

Finally, for rigorous investigations it is better to write a mollified version of the local energy identity, without sharp boundary terms. This is obtained multiplying the pointwise identity above by a smooth test function $\varphi:\mathbb{R}^3\times[0,T]\to\mathbb{R}$, with compact support in the domain of definition of the equations,

$$\varphi\left[\frac{\partial}{\partial t}+u\cdot\nabla-\triangle\right]\left(\frac{1}{2}\left|u\right|^2\right)+\varphi\left\|\nabla u\right\|^2=\varphi u\cdot\left(f-\nabla p\right)$$

and integrating (and using integration by parts):

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^d}\frac{1}{2}\left|u\right|^2\left(\frac{\partial\varphi}{\partial t}+u\cdot\nabla\varphi+\triangle\varphi\right)dxds \qquad (1.9)\\&+\int_0^T\int_{\mathbb{R}^d}\varphi\left\|\nabla u\right\|^2dxds-\int_0^T\int_{\mathbb{R}^d}pu\cdot\nabla\varphi dxds\\&=\int_0^T\int_{\mathbb{R}^d}\varphi u\cdot fdxds.\end{aligned}$$

If we take, in this equation a sequence of test functions of the form $a_n(s)\varphi(x)$ with $a_n(s) \to 1_{[0,t]}(s)$, t given in $(0,T)$, and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ smooth with compact support, since $a_n' \to -\delta_t + \delta_0$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2} |u(t)|^2 \varphi dx + \int_0^t \int_{\mathbb{R}^d} \|\nabla u\|^2 \varphi dx ds \\
&- \int_0^t \int_{\mathbb{R}^d} \frac{1}{2} |u|^2 (u \cdot \nabla \varphi + \triangle \varphi) dx ds \\
&- \int_0^t \int_{\mathbb{R}^d} pu \cdot \nabla \varphi dx ds \\
&= \int_{\mathbb{R}^d} \frac{1}{2} |u(0)|^2 \varphi dx + \int_0^t \int_{\mathbb{R}^d} \varphi u \cdot f dx ds.
\end{aligned}$$

Remark 3 *This equation reduces formally (i.e. for sufficiently regular fields) to the local energy identity (1.7) in a regular set B if we take a sequence $\varphi_n(x) \to 1_B$. Indeed, for smooth vector fields a we have*

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} a \cdot \nabla \varphi_n dx ds &= - \int_0^t \int_{\mathbb{R}^d} \mathtt{div} a \cdot \varphi_n dx ds \\
&\to - \int_0^t \int_B \mathtt{div} a dx ds = \int_0^t \int_{\partial B} a \cdot \nu d\sigma ds
\end{aligned}$$

and for smooth functions a

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} a \cdot \triangle \varphi_n dx ds &= \int_0^t \int_{\mathbb{R}^d} \triangle a \varphi_n dx ds \\
&\to \int_0^t \int_B \triangle a dx ds = \int_0^t \int_{\partial B} \frac{\partial a}{\partial \nu} d\sigma ds.
\end{aligned}$$

1.4 Equation for the pressure

Taking $\mathtt{div}$ of all terms in the Navier-Stokes equations we get

$$\mathtt{div}(u \cdot \nabla u) + \mathtt{div} \nabla p = \mathtt{div} f$$

namely

$$\begin{aligned}\triangle p &= \mathtt{div} f - \sum_{i,k=1}^{3} \partial_i \left(u_k \partial_k u_i\right) \\ &= \mathtt{div} f - \sum_{i,k=1}^{3} \left(\partial_i u_k\right)\left(\partial_k u_i\right) \\ &= \mathtt{div} f - \sum_{i,k=1}^{3} \partial_i \partial_k \left(u_k u_i\right).\end{aligned} \tag{1.10}$$

Sometimes it is better to use the second, sometimes the third identity.

In bounded domain it is quite difficult to use this equation because of boundary conditions. Assume we work in the full space with suitable decay at infinity.

Exercise 4 *Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth compact support function. Show that, around any point $x_0 \in \mathbb{R}^3$,*

$$\begin{aligned}&\int_{\mathbb{R}^3 \setminus B(x_0,\varepsilon)} \frac{1}{\|x - x_0\|} \triangle \varphi\left(x\right) dx \\ &= \int_{\partial B(x_0,\varepsilon)} \left(\frac{1}{\|\sigma - x_0\|} \frac{\partial \varphi}{\partial n} - \frac{(\sigma - x_0) \cdot n}{\|\sigma - x_0\|^3} \varphi \right) d\sigma\end{aligned}$$

where n is the outer normal.

Exercise 5 *Prove that*

$$-\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|x - x_0\|} \triangle \varphi\left(x\right) dx = \varphi\left(x_0\right)$$

for every smooth compact support $\varphi : \mathbb{R}^3 \to \mathbb{R}$.

Clearly the result of these exercises can be extended beyond smooth compact support functions. We thus see that a potentially useful formula to analyse the pressure is

$$p\left(x\right) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|x - y\|} \left(\sum_{i,k=1}^{3} \partial_i \partial_k \left(u_k u_i\right) - \mathtt{div} f \right)\left(y\right) dy. \tag{1.11}$$

The function

$$p(x) + \frac{|v(x)|^2}{2}$$

is physically interesting because in some sense corresponds to an energy (free energy plus kinetic energy). The identity of the next exercise involves similar quantities and localizes the relation between pressure and velocity.

Exercise 6 *(See [7]) Let p and u be related by*

$$p(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|x-y\|} \sum_{i,k=1}^{3} \partial_i \partial_k (u_k u_i)(y)\, dy.$$

Prove that

$$\begin{aligned} &\int_{B(x_0,R)} \frac{1}{\|x-x_0\|} \left(2p(x) + |\widehat{v}^{x_0}(x)|^2\right) dx \\ &= \int_{B(x_0,R)} \frac{1}{R} \left(3p(x) + |v(x)|^2\right) dx \\ &= R^2 \int_{\mathbb{R}^3 \setminus B(x_0,R)} \nabla^2 \frac{1}{\|x-x_0\|} : v(x) \otimes v(x)\, dx \end{aligned}$$

where

$$\widehat{v}^{x_0}(x) = v(x) - \frac{v(x)\cdot(x-x_0)}{\|x-x_0\|^2}(x-x_0)$$

(the orthogonal projection of $v(x)$ into the two-dimensional subspace of $\mathbb{R}^3$ perpendicular to $x-x_0$), ∇^2 gives the matrix of second derivatives, fir a vector a the symbol $a \otimes a$ stands for the matrix of components $a_i a_j$ and the notation $A : B$ between to square matrices stands for the number $A_{ij}B_{ij}$.

Proof. Hint: for every smooth function $\rho : (0,\infty) \to [0,\infty)$,

$$\begin{aligned} &\int_{B(x_0,R)} \rho(\|x-x_0\|)\, p(x)\, dx \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \sum_{i,k=1}^{3} \partial_i \partial_k (u_k u_i)(y) \int_{B(x_0,R)} \frac{1}{\|x-x'\|} \rho(\|x-x_0\|)\, dx dy. \end{aligned}$$

Choose ρ in two different ways. ■

Without details, let us only mention that Fourier analysis is another good way to deal with the Poisson equation (1.10) in the full space with decay at infinity or in the case of periodic boundary conditions. Denoting Fourier transform by $\mathcal{F}$, we have

$$\mathcal{F}\triangle g = -\left|k\right|^2 \mathcal{F}g$$

namely $g = \mathcal{F}^{-1}\left(-\left|k\right|^{-2}\mathcal{F}\triangle g\right)$. Hence

$$p = \mathcal{F}^{-1}\left(-\left|k\right|^{-2}\mathcal{F}\left(\mathtt{div} f - \sum_{i,k=1}^{3}\partial_i\partial_k\left(u_k u_i\right)\right)\right).$$

1.5 Equation for first derivatives

Let us continue to perform formal computations and see which equation is satisfied by the first derivatives $D_i u$ of u. We differentiate in the i-th direction the Navier-Stokes equations:

$$\frac{\partial D_i u}{\partial t} + D_i\left(u\cdot\nabla u\right) + D_i\nabla p = \nu\triangle D_i u + D_i f$$
$$\mathtt{div} D_i u = 0.$$

Since $D_i\left(u\cdot\nabla u\right) = u\cdot\nabla D_i u + D_i u\cdot\nabla u$, we get

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla - \nu\triangle\right) D_i u = -\nabla D_i p + D_i f - D_i u\cdot\nabla u.$$

We may consider this as a transport-diffusion equation of the form

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta - \triangle\theta = g.$$

There is a problem with boundary conditions since we have no natural conditions on $D_i u$ on the boundary. Thus let us restrict ourselves to the full space.

Let us see two typical tricks of PDEs at work. The first one is to multiply by the unknown function, integrate in space and integrate by parts, as we did above to get the energy identity. Here we find

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|D_i u\right|^2 dx + \nu\int_{\mathbb{R}^3}\left|\nabla D_i u\right|^2 dx = \int_{\mathbb{R}^3}\left(D_i f - D_i u\cdot\nabla u\right)\cdot D_i u dx$$
$$= \int_{\mathbb{R}^3} D_i f\cdot D_i u dx + \int_{\mathbb{R}^3}\left(u\cdot\nabla u\right) D_i^2 u dx$$

$$\leq \int_{\mathbb{R}^3} |D_i f|^2 \, dx + \int_{\mathbb{R}^3} |D_i u|^2 \, dx + \frac{\nu}{2} \int_{\mathbb{R}^3} \left|D_i^2 u\right|^2 dx + C_\nu \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dx.$$

As we shall explain better in other chapters, we have the inequalities

$$\begin{aligned}\int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dx &\leq C \left(\int_{\mathbb{R}^3} |u|^3 \, dx\right)^{2/3} \left(\int_{\mathbb{R}^3} |\nabla u|^6 \, dx\right)^{1/3} \\ &\leq C \left(\int_{\mathbb{R}^3} |u|^3 \, dx\right)^{2/3} \int_{\mathbb{R}^3} \left|\nabla^2 u\right|^2 dx\end{aligned}$$

because $W^{1,2} \subset L^6$. We sum over i and get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |Du|^2 \, dx + \left(\frac{\nu}{2} - C\left(\int_{\mathbb{R}^3} |u|^3 \, dx\right)^{2/3}\right) \int_{\mathbb{R}^3} \left|\nabla^2 u\right|^2 dx \\ &\leq \int_{\mathbb{R}^3} |D_i f|^2 \, dx + \int_{\mathbb{R}^3} |D_i u|^2 \, dx.\end{aligned}$$

If

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx$$

is sufficiently small, then $\frac{\nu}{2} - C\left(\int_{\mathbb{R}^3} |u|^3 \, dx\right)^{2/3} > 0$ and, assuming $\int_{\mathbb{R}^3} |\nabla u_0(x)|^2 \, dx < \infty$, we get from Gronwall lemma

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} \left|\nabla^2 u\right|^2 dx dt < \infty. \tag{1.12}$$

In other words, if $\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx$ is small enough and the initial condition is regular, these heuristic computations show that the solution should be more regular than (1.3). Of course, a priori, there is no reason for having $\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx$ small. Much worse, there is no reason to have

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx < \infty. \tag{1.13}$$

So we do not know whether (1.12) is a property of all solutions; we have it only for those with small $L^\infty(0,T;L^3)$-norm.

The difference between a regularity conditions of the form (1.3) and (1.12) is enormous in terms of the theory of 3D Navier-Stokes equations. Under condition (1.3) we do not know if solutions are unique (see section 1.2). On

the contrary, under the regularity condition (1.12) the solution is unique and and does not develop singularities, if the data are more regular.

As an example of simple computation, let us show that (1.12) implies (1.4), thus uniqueness, at least heuristically as in the general style of this chapter. We have

$$\left| \int_0^t \int_{\mathbb{R}^3} v \cdot \left(v \cdot \nabla u^{(2)} \right) dxds \right| = \left| \int_0^t \int_{\mathbb{R}^3} u^{(2)} \cdot \left(v \cdot \nabla v \right) dxds \right|$$

by integration by parts. Since $W^{2,2}\left(\mathbb{R}^3\right) \subset C^{0,1/2}\left(\mathbb{R}^3\right)$ (see the appendix), we have in particular

$$\left\| u^{(2)}(t) \right\|_\infty^2 := \sup_{x \in \mathbb{R}^3} \left| u^{(2)}(x,t) \right|^2 \leq C \int_{\mathbb{R}^3} \left| \nabla^2 u^{(2)} \right|^2 dx \in L^1(0,T)$$

Hence

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}^3} v \cdot \left(v \cdot \nabla u^{(2)} \right) dxds \right| \\ &\leq C \int_0^t \left\| u^{(2)}(s) \right\|_\infty \int_{\mathbb{R}^3} |v| \, \|\nabla v\| \, dxds \\ &\leq \frac{\nu}{2} \int_0^t \int_{\mathbb{R}^3} \|\nabla v\|^2 \, dxds + C_\nu \int_0^t \left\| u^{(2)}(s) \right\|_\infty \int_{\mathbb{R}^3} |v|^2 \, dxds \end{aligned}$$

and (1.4) is proved with $\theta^{(2)}(t) = \left\| u^{(2)}(t) \right\|_\infty^2$.

The previous discussion is an example of easy computations but it may be misleading in terms of what is known or not. With less easy proofs one can see that (1.13) implies uniqueness (directly, namely without going through property (1.12)). But even an increase of regularity is true under (1.13), see [8]. Let us stress that (1.13) is unproved. The proof of [8] is very long. If in addition to (1.13) we ask a little bit more, in the direction of the continuity in time of $\int |u(x,t)|^3 \, dx$, a transparent proof along the lines just described above has been given by [2].

A second well known method for transport-diffusion equation is related to the concept of renormalized solution. If $\beta : \mathbb{R} \to \mathbb{R}$ is a smooth function,

then

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)\beta\left(D_iu_j\right)\\
&=\beta'\left(D_iu\right)\cdot\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)D_iu_j\\
&-\sum_k\beta''\left|D_kD_iu_j\right|^2\\
&=\beta'\left(D_iu\right)\left(-D_jD_ip+D_if_j-D_iu\cdot\nabla u_j\right)\\
&-\sum_k\beta''\left|D_kD_iu_j\right|^2
\end{aligned}$$

because

$$\begin{aligned}
\triangle\beta\left(D_iu_j\right)&=\sum_kD_k\left(\beta'D_kD_iu_j\right)\\
&=\sum_k\beta''\left|D_kD_iu_j\right|^2+\sum_k\beta'D_k^2D_iu_j.
\end{aligned}$$

The choice $\beta\left(r\right)=r^2$ gives the result above. It is interesting for speculation but requires at the end unproved assumptions, as we have seen. On the contrary, let us take

$$\beta\left(r\right)=\left|r\right|$$

(we should use a convex smooth approximation of the absolute value, to be more rigorous). From the convexity of β and the boundedness of $\beta'\left(r\right)=\frac{r}{|r|}$ we have

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)\left|D_iu_j\right|\leq\left|-D_jD_ip+D_if_j-D_iu\cdot\nabla u_j\right|$$

namely

$$\begin{aligned}
\int_{\mathbb{R}^3}\left|D_iu_j\left(x,t\right)\right|dx&\leq\int_{\mathbb{R}^3}\left|D_iu_j\left(x,0\right)\right|dx\\
&+\int_0^t\int_{\mathbb{R}^3}\left(\left|D_jD_ip\right|+\left|D_if_j\right|+\left|\nabla u\right|^2\right)dxds.
\end{aligned}$$

Again we have assumed to work in the full space. Assume initial condition and f sufficiently regular. Recall that $\left|\nabla u\right|^2$ is integrable, see (1.3). If we

prove that $|D_j D_i p|$ is also integrable, then we obtain

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |D_i u_j (x,t)|\, dx < \infty. \tag{1.14}$$

The function $|D_j D_i p|$ is really integrable, but this is a nontrivial result that we cannot show in this introduction. Modulo the proof of such a delicate fact, the estimate (1.14) is true. In the next section we show a similar result for the vorticity that is not based on difficult properties of the pressure. The striking fact is that estimate (1.14) is true for all weak solutions of 3D Navier-Stokes equations: it is not a conditional result as (1.12). Unfortunately, it seems that it does not add too much, to the point that until now nobody introduced it in the definition itself of weak solution. For instance, Sobolev embedding does not improve (1.3). But on the example developed in section 1.2 we have

$$\begin{aligned}\int_{\mathbb{R}^3} \|\nabla u (x,t)\|\, dx &= \alpha^{-1/2}(t) \int_{\mathbb{R}^3} |(\nabla\varphi)(x\alpha(t))|\, \alpha^3(t)\, dx \\ &= \alpha^{-1/2}(t) \int_{\mathbb{R}^3} \|\nabla\varphi\|^2\, dx\end{aligned}$$

hence (1.14) implies that we have to impose

$$\alpha(t) \geq \alpha_0 > 0.$$

This constraint did not emerge from (1.3) and may have a deep meaning.

Bernstein method is a variant of the previous ideas but looks less suitable in this context. Another variant of interest has been developed by Foias but will be described elsewhere.

Remark 7 *Since we shall deal with stochastic processes in the sequel, let us mention that the solution of the transport-diffusion equation*

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta - \triangle\theta = g$$

has the following probabilistic representation

$$\theta(t,x) = E\left[\theta\left(0, X_0^{t,x}\right)\right] + \int_0^t E\left[g\left(s, X_s^{t,x}\right)\right] ds$$

for a suitable random field $X_s^{t,x}$ such that $x \mapsto X_s^{t,x}$ is measure preserving. If g is L^1 we have

$$\begin{aligned}\int |\theta(t,x)|\,dx &\leq E\int \left|\theta\left(0,X_0^{t,x}\right)\right|dx + \int_0^t E\left[\int \left|g\left(s,X_s^{t,x}\right)\right|dx\right]ds \\ &= E\int |\theta(0,y)|\,dy + \int_0^t E\left[\int |g(s,y)|\,dy\right]ds\end{aligned}$$

hence

$$\sup_{t\in[0,T]} \|\theta(t,\cdot)\|_{L^1(\mathbb{R}^3)} \leq \|\theta(0,\cdot)\|_{L^1(\mathbb{R}^3)} + \|g\|_{L^1([0,T]\times\mathbb{R}^3)}.$$

This is another way to understand the result obtained above by the function $\beta(r) = |r|$.

1.6 Equation for the vorticity

Assume we work in the full space $\mathbb{R}^3$. By `curl` we mean the differential operator on vector fields $u : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\texttt{curl}u = (\partial_3 u_2 - \partial_2 u_3, \partial_1 u_3 - \partial_3 u_1, \partial_2 u_1 - \partial_1 u_2).$$

We have

$$\begin{aligned}\texttt{curl}\nabla q &= (\partial_3\partial_2 - \partial_2\partial_3, \partial_1\partial_3 - \partial_3\partial_1, \partial_2\partial_1 - \partial_1\partial_2)\, q \\ &= 0\end{aligned}$$

on smooth fields $q : \mathbb{R}^3 \to \mathbb{R}$ and thus by continuity on several classes where smooth fields are dense. Let us apply `curl` to all terms of the Navier-Stokes equations and denote `curl`u by ξ:

$$\frac{\partial \xi}{\partial t} + \texttt{curl}\,(u\cdot\nabla u) = \nu\triangle\xi + \texttt{curl}f.$$

Exercise 8 *Prove that*

$$\texttt{curl}\,(u\cdot\nabla u) = u\cdot\nabla\xi - \xi\cdot\nabla u.$$

The formula in the exercise yields

$$\frac{\partial \xi}{\partial t} + u\cdot\nabla\xi = \nu\triangle\xi + \xi\cdot\nabla u + \texttt{curl}f.$$

The vector field ξ is called *vorticity* field, and this equation is the vorticity form of the Navier-Stokes equations. The pressure is disappeared.

The interpretation of the vorticity field may come from Taylor development. We know that

$$u(x) = u(x_0) + (x - x_0) \cdot \nabla u(x_0) + o(\|x - x_0\|)$$

(for differentiable fields). We can rewrite the matrix $\nabla u(x_0)$ as

$$\nabla u(x_0) = \frac{\nabla u(x_0) + \nabla u(x_0)^T}{2} + \frac{\nabla u(x_0) - \nabla u(x_0)^T}{2}$$

and call $S_u(x_0)$ the first term, $\Omega_u(x_0)$ the second one. $S_u(x_0)$ is called *deformation tensor*. It has zero trace (its trace is the divergence of u). Being $(\nabla u)_{ij} = \partial_i u_j$, we see that

$$\texttt{curl} u = 2\left((\Omega_u)_{3,2}, (\Omega_u)_{1,3}, (\Omega_u)_{2,1}\right)$$

or

$$\Omega_u = \frac{1}{2}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Hence

$$(x - x_0) \cdot \Omega_u(x_0) = (x - x_0) \wedge \xi(x_0).$$

A vector field purely of the form

$$v(x) := (x - x_0) \wedge \xi_0$$

is a rotation around the ξ_0 axis. Summarizing,

$$u(x) = u(x_0) + (x - x_0) \cdot S_u(x_0) + (x - x_0) \wedge \xi(x_0) + o(\|x - x_0\|)$$

which means that any field can be seen as the superposition of the action of a zero-trace symmetric matrix (a deformation) plus a rotation around the vorticity field.

Since $\xi \wedge \xi = 0$,

$$\xi \cdot \nabla u = \xi \cdot S_u$$

and thus we can rewrite the vorticity equation in the form

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = \nu \triangle \xi + \xi \cdot S_u + \texttt{curl} f.$$

The term $\xi \cdot S_u$ is called *vortex stretching* term because it describes the action on the vorticity field (for instance the elongation) due to the deformation tensor. The vortex stretching term is in a sense the main source of difficulties in the theoretical analysis of the 3D-Navier-Stokes equations. One could develop the following picture (we do not know to which extent it is true): vorticity is not just transported and diffused, but may increase by stretching; higher vorticity may imply higher velocities locally around the axis of rotation; this may produce more intense deformation tensors and increase the vorticity further. Maybe this mechanism may blow up under special geometric circumstances.

The identity $\xi = \mathtt{curl} u$ can be inverted. Thanks to the condition $\mathtt{div} u = 0$, we have

$$\mathtt{curlcurl}\varphi = -\triangle\varphi$$

because, componentwise,

$$\begin{aligned}
&\partial_3\left(\partial_1\varphi_3 - \partial_3\varphi_1\right) - \partial_2\left(\partial_2\varphi_1 - \partial_1\varphi_2\right) \\
&= \partial_1\left(\partial_3\varphi_3 + \partial_2\varphi_2\right) - \left(\partial_3\partial_3 + \partial_2\partial_2\right)\varphi_1 \\
&= -\partial_1\left(\partial_1\varphi_1\right) - \left(\partial_3\partial_3 + \partial_2\partial_2\right)\varphi_1 = \triangle\varphi_1
\end{aligned}$$

having used the condition $\mathtt{div} u = 0$, and so on for the other components. Then solve the Poisson equation $\triangle\varphi = -\xi$ and set

$$u = -\mathtt{curl}\triangle^{-1}\xi.$$

We see that $\mathtt{curl} u = \xi$. One can argue that this is the only divergence free field such that $\mathtt{curl} u = \xi$. This equation is called the *Biot-Savart law*. Summarizing, the vorticity equation is a closed equation for the vorticity field if we substitute $u = -\mathtt{curl}\triangle^{-1}\xi$ but the operator $\triangle^{-1}$ is non-local.

In dimension 3 the vorticity equation is rarely used because of the vortex stretching term. But some very advanced results have been proved based on it. Let us explicit the balance law for the quantity $\int_{\mathbb{R}^3} |\xi|^2 \, dx$ called *enstrophy*. Multiplying by ξ, integrating and using integration by parts and $\mathtt{div} u = 0$ we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\xi|^2 \, dx + \nu \int_{\mathbb{R}^3} \|\nabla\xi\|^2 \, dx = \int_{\mathbb{R}^3} \xi^T S_u \xi dx + \int_{\mathbb{R}^3} \xi \cdot \mathtt{curl} f dx.$$

This identity states that the variation of enstrophy is due to friction, vorticity injected by the body force f, but also by possible vortex stretching.

If we look at the vorticity equation

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)\xi=\xi\cdot\nabla u+\mathtt{curl}f$$

as a transport-diffusion equation and we apply the method of renormalized solutions described in the previous section, with $\beta(r)=|r|$ we get

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)|\xi_j|\leq\left|\xi\cdot\nabla u_j+(\mathtt{curl}f)_j\right|$$

hence

$$\begin{aligned}\int_{\mathbb{R}^3}|\xi_j(x,t)|\,dx\leq\int_{\mathbb{R}^3}|\xi(x,0)|\,dx+C\int_0^t\int_{\mathbb{R}^3}\|\nabla u\|^2\,dxds\\+\int_0^t\int_{\mathbb{R}^3}|\mathtt{curl}f|\,dxds\end{aligned}$$

because we can bound $|\xi|$ and $|\nabla u_j|$ by $C\|\nabla u\|^2$. Thus, if

$$\mathtt{curl}u_0\in L^1\left(\mathbb{R}^3\right),\quad\int_0^T\int_{\mathbb{R}^3}|\mathtt{curl}f|\,dxds<\infty$$

we get

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}|\mathtt{curl}u(x,t)|\,dx<\infty.\tag{1.15}$$

This is slightly weaker that property (1.14), but still very interesting and its (heuristic) proof has been elementary.

1.7 Two dimensional fluids

A fluid is two-dimensional, in a strict sense, if $u=(u_1,u_2,0)$, with u_1 and u_2 independent of x_3. Strictly speaking, 2D fluids do not exist, but some relevant physical examples can be considered two-dimensional at proper scales (like the atmosphere), or at least the 2D idealization is a first numerical and theoretical step. These lectures are not devoted to 2D fluids, but it is important to mention the main theoretical differences. Let us mention two of them: weak solutions are unique; the vorticity equation does not contain the vortex stretching term.

To see uniqueness of solutions with the degree of regularity (1.3) we have to prove (1.4). Assume we work in a domain D (also the full space) with suitable conditions on the boundary. We have

$$\begin{aligned}&\left|\int_0^t\int_D v\cdot\left(v\cdot\nabla u^{(2)}\right)dxds\right|\\&\leq\int_0^t\left(\int_D\left\|\nabla u^{(2)}\right\|^2dx\right)^{1/2}\left(\int_D|v|^4\,dx\right)^{1/2}ds\end{aligned}$$

and since by Ladyzhenskaya multiplicative inequality (see the appendix)

$$\left(\int_D|v|^4\,dx\right)^{1/4}\leq C\left(\int_D|v|^2\,dx\right)^{1/4}\left(\int_D|\nabla v|^2\,dx\right)^{1/4}$$

we get

$$\begin{aligned}&\leq C\int_0^t\left(\int_D\left\|\nabla u^{(2)}\right\|^2dx\right)^{1/2}\left(\int_D|v|^2\,dx\right)^{1/2}\left(\int_D|\nabla v|^2\,dx\right)^{1/2}ds\\&\leq\frac{\nu}{2}\int_0^t\int_D|\nabla v|^2\,dxds+C_\nu\int_0^t\int_D\left\|\nabla u^{(2)}\right\|^2dx\int_D|v|^2\,dxds\end{aligned}$$

so (1.4) is true with

$$\theta^{(2)}(t)=\int_D\left\|\nabla u^{(2)}(x,t)\right\|^2dx\in L^1(0,T).$$

About the vorticity equation, notice that $\xi=(0,0,\partial_2u_1-\partial_1u_2)$. The vorticity is then described by the scalar field

$$\widetilde{\xi}=\partial_2u_1-\partial_1u_2$$

denoted also by $\nabla^\perp u$. The difficult term $\xi\cdot\nabla u$ disappears since ξ has only the third component different from zero and ∂_3u is zero. Thus the vorticity equation is

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla-\nu\triangle\right)\xi=\mathtt{curl}f.$$

The absence of vortex stretching means that vorticity is just transported and diffused. This makes impossible the potential mechanism of blow-up due to vortex stretching.

Since we do not have the unknown on the right-hand-side, but just the data, when the data are sufficiently regular we may simply deduce a corresponding regularity of ξ, a fact impossible in 3D. For instance,

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^2}\left|\xi\left(x,t\right)\right|^2 dx \leq \int_{\mathbb{R}^2}\left|\mathtt{curl}u_0\right|^2 dx + \int_0^T\int_{\mathbb{R}^2}\left|\mathtt{curl}f\right|^2 dxds.$$

We can even apply the maximum principle and obtain

$$\sup_{(x,t)}\left|\xi\left(x,t\right)\right| \leq \sup_{x}\left|\mathtt{curl}u_0\left(x\right)\right| + T\sup_{(x,t)}\left|\mathtt{curl}f\left(x,t\right)\right|.$$

These results are remarkable also because they are independent of the viscosity, thus they may help in the analysis of the limit $\nu \to 0$, and they should survive for the Euler equation. Indeed, the theory of the 2D Euler equation contains good results, in particular the global existence when $\int_{\mathbb{R}^2}\left|\mathtt{curl}u_0\right|^2 dx < \infty$ (thanks to the first estimate above) and the uniqueness when $\sup_x\left|\mathtt{curl}u_0\left(x\right)\right| < \infty$ (related to the second estimate above, but the argument is more difficult). This is not the case in 3D, where there is not even a global existence result.

Remark 9 *The maximum principle can be proved by means of analytic techniques, but it is particularly evident from the probabilistic representation*

$$\xi\left(t,x\right) = E\left[\xi\left(0,X_0^{t,x}\right)\right] + \int_0^t E\left[\mathtt{curl}f\left(s,X_s^{t,x}\right)\right]ds$$

introduced in remark 7 above.

1.8 Appendix

1.8.1 Gauss-Green formula

If $D \subset \mathbb{R}^d$ is a bounded open set with smooth boundary, f, g are smooth functions, then

$$\int_D D_k f\left(x\right) g\left(x\right) dx = -\int_D f\left(x\right) D_k g\left(x\right) dx + \int_{\partial D} f\left(\sigma\right) g\left(\sigma\right) n_k\left(\sigma\right) d\sigma.$$

In particular, if v is a smooth field, then

$$\int_D \nabla f\left(x\right)\cdot v\left(x\right) dx = -\int_D f\left(x\right)\mathtt{div}v\left(x\right) dx + \int_{\partial D} f\left(\sigma\right) v\left(\sigma\right)\cdot n\left(\sigma\right) d\sigma.$$

These identities extend to Lipschitz boundary, functions and fields in Sobolev spaces, for all exponents such that the integrals are well defined.

1.8.2 Gronwall lemma

If $a(t)$, $t \in [0,T]$, is a positive measurable function, $b(t)$ is positive integrable function and

$$a(t) \leq a(0) + \int_0^t b(s)a(s)ds \quad \text{for } t \in [0,T]$$

then

$$a(t) \leq a(0)e^{\int_0^t b(s)ds} \quad \text{for } t \in [0,T].$$

1.8.3 Function spaces

Given a measurable set $D \subset \mathbb{R}^d$ and a number $p \geq 1$, $L^p(D)$ denotes the space of all measurable functions $f : D \to \mathbb{R}$ such that $\int_D |f(x)|^p dx < \infty$. $L^p(D)$ is a Banach space with the norm

$$\|f\|_{L^p} = \int_D |f(x)|^p dx.$$

Hölder inequality states that

$$\int_D |f(x)\, g(x)|\, dx \leq \left(\int_D |f(x)|^p dx\right)^{1/p} \left(\int_D |g(x)|^q dx\right)^{1/q}$$

for $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Hence, if D is bounded, $L^{p_1}(D) \subset L^{p_2}(D)$ for $p_1 \geq p_2$; if D is not bounded, we can say that $L^{p_1}(D) \cap L^{p_2}(D) \subset L^p(D)$ for all $p \in [p_1, p_2]$.

We denote by $L^p_{loc}(D)$ the space of all measurable functions $f : D \to \mathbb{R}$ such that $f|_{D'} \in L^p(D')$ for all bounded set $D' \subset D$.

Let D be an open set. We denote by $C_0^\infty(D)$ the space of all infinitely differentiable functions $f : D \to \mathbb{R}$ with compact support in D.

We say that a function $f \in L^1_{loc}(D)$ has the weak derivative $\partial_k f$ in $L^p(D)$ if there is a function $g \in L^p(D)$ such that

$$\int_D g(x)\, \varphi(x)\, dx = -\int_D f(x)\, \partial_k \varphi(x)\, dx$$

for all $\varphi \in C_0^\infty(D)$. We use the symbol $\partial_k f$ for the function g (which is proved to be unique).

We denote by $W^{1,p}(D)$ the space of all $f \in L^p(D)$ with $\partial_k f \in L^p(D)$ for $k = 1, ..., d$. $W^{1,p}(D)$ is a Banach space with the norm

$$\|f\|_{W^{1,p}} = \int_D |f(x)|^p \, dx + \sum_{k=}^{d} \int_D |\partial_k f(x)|^p \, dx.$$

We denote by $W_0^{1,p}(D)$ the closure in $W^{1,p}(D)$ of $C_0^\infty(D)$. We have $W_0^{1,p}\left(\mathbb{R}^d\right) = W^{1,p}\left(\mathbb{R}^d\right)$.

Given $\alpha \in (0,1)$, we denote by $C^{0,\alpha}(D)$ the space of all bounded continuous functions $f : D \to \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M \|x - y\|^\alpha, \quad x, y \in D$$

for some constant $M > 0$. $C^{0,\alpha}(D)$ is a Banach space with the norm

$$\|f\|_{C^{0,\alpha}} = \sup_{x \in D} |f(x)| + \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}.$$

1.8.4 Embeddings and inequalities

Details on the topics of this section can be found in [1], [3], [9] and several other books and papers.

Given two Banach spaces $\left(B_i, \|.\|_{B_i}\right)$, $i = 1, 2$, we say that B_1 is contained in B_2 with continuous embedding, and write $B_1 \subset B_2$, if every element of B_1 is in B_2 and there is a constant $C > 0$ such that

$$\|x\|_{B_2} \leq C \|x\|_{B_1}$$

for all $x \in B_1$.

Given the dimension d, for every $p \geq 1$ set

$$p^* = \frac{dp}{d - p}, \quad \alpha = 1 - \frac{d}{p}.$$

In $\mathbb{R}^d$, Sobolev embedding theorem states that

$$\begin{aligned} 1 \leq p < d &\Rightarrow W^{1,p}\left(\mathbb{R}^d\right) \subset L^{p^*}\left(\mathbb{R}^d\right) \\ p > d &\Rightarrow W^{1,p}\left(\mathbb{R}^d\right) \subset C^{0,\alpha}\left(\mathbb{R}^d\right) \\ W^{1,d}\left(\mathbb{R}^d\right) &\subset L^q\left(\mathbb{R}^d\right) \quad \text{for every } q \in [d, \infty). \end{aligned}$$

In a bounded open domain D with Lipschitz boundary, we have a similar result:

$$1 \leq p < d \Rightarrow W^{1,p}(D) \subset L^{p^*}(D)$$
$$p > d \Rightarrow W^{1,p}(D) \subset C^{0,\alpha}\left(\overline{D}\right)$$
$$W^{1,d}(D) \subset L^q(D) \quad \text{for every } q \in [1,\infty).$$

The inequalities corresponding to these inclusions have the form

$$\|f\|_{L^{p^*}} \leq C \|f\|_{W^{1,p}} \text{ for } 1 \leq p < d$$

and so on. We stress that on the right-hand-side we have the full $W^{1,p}$-norm.

For any open set D, if we restrict the attention to $f \in W_0^{1,p}(D)$, there exist a few stronger inequalities. Notice that if $D = \mathbb{R}^d$ this is true for all $f \in W^{1,p}(D)$. First, for $1 \leq p < d$

$$\left(\int_D |f(x)|^{p^*} dx\right)^{1/p^*} \leq C \left(\int_D |\nabla f(x)|^p dx\right)^{1/p} \tag{1.16}$$

but we do not have a useful estimate for $p = d$. Second, we also have the so-called *multiplicative inequalities*

$$\left(\int_D |f(x)|^r dx\right)^{1/r} \leq C \left(\int_D |f(x)|^p dx\right)^{\frac{1-\lambda}{p}} \left(\int_D |\nabla f(x)|^p dx\right)^{\frac{\lambda}{p}} \tag{1.17}$$
$$\lambda = \frac{d(r-p)}{rp}$$

valid either if

$$1 \leq p < d, \quad r \in [p, p^*]$$

or if

$$p \geq d, \quad r \geq p.$$

A remarkable example is Ladyzhenskaya inequality [5]

$$\left(\int_{\mathbb{R}^2} |f(x)|^4 dx\right)^{1/4} \leq C \left(\int_{\mathbb{R}^2} |f(x)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^2} |\nabla f(x)|^2 dx\right)^{1/2}$$

which can be used to prove uniqueness of weak solutions in 2D, while in 3D we only have

$$\left(\int_{\mathbb{R}^3} |f(x)|^4 dx\right)^{1/4} \leq C \left(\int_{\mathbb{R}^3} |f(x)|^2 dx\right)^{1/4} \left(\int_{\mathbb{R}^3} |\nabla f(x)|^2 dx\right)^{3/4}.$$

Exercise 10 *Prove in some case (1.17) from (1.16).*

If we consider a bounded domain with Lipschitz boundary and functions of class $W^{1,p}(D)$ instead of $W_0^{1,p}(D)$, we have seen above that Sobolev embedding is true, just we need to use the full norm $\|f\|_{W^{1,p}}$ on the right-hand-side of the inequalities. This is true also for the multiplicative inequality (just repeat the previous exercise):

$$\|f\|_{L^r} \leq C_D \|f\|_{L^p}^{1-\lambda} \|f\|_{W^{1,p}}^{\lambda} \qquad \text{for all } f \in W^{1,p}(D)$$

always with $1 \leq p < d$, $r \in [p, p^*]$, $\lambda = \frac{d(r-p)}{rp}$. The constant C_D depends now on the domain.

Thus, for instance, we may prove the following local version of the multiplicative inequality.

Exercise 11 *With the notation $B_R = B(x_0, R)$, in dimension 3 prove that*

$$\int_{B_R} |f(x)|^r \, dx \leq C \left(\int_{B_R} |f(x)|^2 \, dx \right)^{\frac{r}{2}-a} \left(\int_{B_R} |\nabla f(x)|^2 \, dx \right)^a + \frac{C}{R^{2a}} \left(\int_{B_R} |f(x)|^2 \, dx \right)^{\frac{r}{2}}$$
$$r \in [2, 6], \quad a = \frac{3}{4}(r-2).$$

Proof. To obtain the factor $\frac{1}{R^{2a}}$, which is related to the constant C_D mentioned above, rescale B_R to $B(x_0, 1)$ and use the constant $C_{B(x_0,1)}$. ■

There is another way to understand or deduce the multiplicative inequalities, by means of Sobolev embedding with fractional order Sobolev spaces, followed by interpolation inequalities, see for instance [9].

Let us finally mention that Sobolev inequalities can be iterated to higher derivatives. For instance, we know that

$$W^{2,p}(\mathbb{R}^d) \subset W^{1,p^*}(\mathbb{R}^d), \quad p^* = \frac{dp}{d-p}$$

and if $p^* > d$ we have

$$W^{1,p^*}(\mathbb{R}^d) \subset C^{0,\alpha^*}(\mathbb{R}^d), \quad \alpha^* = 1 - \frac{d}{p^*}.$$

Then

$$p > \frac{d}{2} \Rightarrow W^{2,p}\left(\mathbb{R}^d\right) \subset C^{0,\alpha^*}\left(\mathbb{R}^d\right), \quad \alpha^* = 2 - \frac{d}{p}.$$

In particular,

$$W^{2,2}\left(\mathbb{R}^3\right) \subset C^{0,\frac{1}{2}}\left(\mathbb{R}^3\right).$$

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Chapter 2

Rigorous results for periodic fields

2.1 Introduction

In this section we consider the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \triangle u + f$$
$$\mathtt{div} u = 0$$
$$u|_{t=0} = u_0$$

in $\mathbb{R}^3 \times [0,T]$, but we suppose that u_0 and f are L-periodic measurable functions of x, L-periodic in all directions, namely

$$u_0\left(x + \underline{k}L\right) = u_0\left(x\right) \quad \text{for a.e. } x \in \mathbb{R}^3$$
$$f\left(x + \underline{k}L, t\right) = f(x,t) \quad \text{for a.e. } (x,t) \in \mathbb{R}^3 \times [0,T]$$

for all $\underline{k} \in \mathbb{Z}^d$ and we look for L-periodic solutions (u,p). Of course the case of different periods in different directions can be re-scaled to this one.

With different language we could say that we study the equation in the domain $D = [0,L]^3$ with periodic boundary conditions. Or in other words that we study the equation on the *torus* of size L. However, the concept of boundary conditions for not so smooth fields is somewhat delicate and thus it is easier to think that we work on the full space but the fields are periodic. Technically, one difference between these two viewpoints is the meaning of

the condition

$$\mathtt{div} v = 0$$

imposed on (at least locally integrable) vector fields v. Working in the full space, we understand it in the sense of distribution on $\mathbb{R}^3$, namely

$$\int_{\mathbb{R}^3} v \cdot \nabla \varphi dx = 0$$

for all smooth compact support test functions $\varphi : \mathbb{R}^3 \to \mathbb{R}$. This condition plus the periodicity implicitly imposes some kind of weak periodicity condition on the boundary (see the appendix A). On the contrary, if we choose to work on $[0, L]^3$ with periodic boundary conditions, we understand $\mathtt{div} v = 0$ in the sense of distribution on $[0, L]^3$, namely

$$\int_{[0,L]^3} v \cdot \nabla \varphi dx = 0$$

for all smooth compact support test functions $\varphi : (0, L)^3 \to \mathbb{R}$. In this case some periodicity of v on the boundary must be imposed explicitly. We thus prefer the approach in the full space.

From the physical viewpoint the periodic problem is artificial but in view of the extreme difficulty of the 3D Navier-Stokes equations it is meaningful to analyze in detail this model problem, free of some technical difficulties due to non-slip boundary conditions or non-compactness in the full space. Let us recall that the periodic case is widely investigated in the physical literature too (see for instance [5]) and is one version of the millennium problem for the 3D Navier-Stokes equations (see [3]). So, at least for theoretical mathematics and theoretical physics it is a respectable problem.

2.2 Function spaces

Recall that heuristic computations show that property (2.1) below is almost the only quantitative information one knows for solutions of 3D Navier-Stokes equations. Therefore it is natural to consider the following two functions spaces of space-vector fields, appearing in (2.1).

Preliminary, we denote by $C^\infty_{0,per,div}(\mathbb{R}^3, \mathbb{R}^3)$ the space of all smooth L-periodic divergence free vector fields $v : \mathbb{R}^3 \to \mathbb{R}^3$. We denote by $\mathcal{D}$ the

subspace of all $v \in C^{\infty}_{0,per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ such that

$$\int_{[0,L]^3} v\left(x\right) dx = 0.$$

We denote by $L^2_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ the space of all measurable L-periodic vector fields $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\int_{[0,L]^3} \left|v\left(x\right)\right|^2 dx < \infty$$

and $\mathtt{div} v = 0$ in the sense of distributions on $\mathbb{R}^3$ (as explained above or in appendix A). We denote by H the subspace of $L^2_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ defined by the additional condition $\int_{[0,L]^3} v\left(x\right) dx = 0$. The spaces $L^2_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ and H are Hilbert spaces with the scalar product

$$\left\langle u,v\right\rangle_H := \int_{[0,L]^3} u\left(x\right)\cdot v\left(x\right) dx.$$

We denote the norm in this spaces by $\left|.\right|_H$, $\left|u\right|^2_H := \left\langle u,u\right\rangle_H$.

We denote by $H^1_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ the subspace of $L^2_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ of all fields $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that for the first distributional derivatives we have $\partial_i v_j \in L^2\left(\left[0,L\right]^3\right)$ for all $i,j = 1,2,3$ (see also the appendix of Chapter 1). We denote by V the subspace of $H^1_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ defined by the additional condition $\int_{[0,L]^3} v\left(x\right) dx = 0$. The space $H^1_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)$ is a Hilbert space with the scalar product

$$\left\langle u,v\right\rangle_H + \left\langle u,v\right\rangle_V$$

where we set

$$\begin{aligned}\left\langle u,v\right\rangle_V &= \int_{[0,L]^3} \nabla u\left(x\right) : \nabla v\left(x\right) dx \\ &= \sum_{ij=1}^{3} \int_{[0,L]^3} \partial_i u_j\left(x\right) \partial_i v_j\left(x\right) dx.\end{aligned}$$

We have used also the not always common but useful notation

$$A : B = \sum_{ij=1}^{d} A_{ij} B_{ij}$$

when A and B are $d \times d$-matrices.

Put $\|u\|_V^2 := \langle u, u \rangle_V$. Also the space V is Hilbert with the scalar product $\langle u, v \rangle_H + \langle u, v \rangle_V$ but the associated norm $\sqrt{|.|_H^2 + \|.\|_V^2}$ is equivalent to $\|.\|_V$ alone, because there exists a constant $\lambda > 0$, called Poincaré constant, such that

$$|u|_H^2 \leq \lambda \|u\|_V^2 \quad \text{for all } u \in V.$$

We also have

$$\lambda = \left(\frac{L}{2\pi} \right)^2.$$

This is not true in $H^1_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$: non-zero constant fields do not satisfy it. The additional constraint $\int_{[0,L]^3} u(x)\, dx = 0$ makes it true.

Exercise 12 *Prove Poincaré inequality above by Fourier analysis (see appendix B for details on Fourier analysis in these spaces).*

In the sequel, we shall always endow V with the norm $\|.\|_V$.

As explained in appendix A, the boundary value of a field $v \in H^1_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ on $\partial [0, L]^3$ is a well defined square integrable field (even more regular); from L-periodicity it follows that $v|_{\partial [0,L]^3}$ is equal on opposite sides of $[0, L]^3$. On the contrary, the boundary value $v|_{\partial [0,L]^3}$ is not well defined if v is only of class $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$, but at least an object corresponding to $v|_{\partial [0,L]^3} \cdot n$ is well defined in the sense of distributions. It must be equal (as localized distributions) on opposite sides of $[0, L]^3$. This is explained in appendix A.

Given $v \in L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ define

$$v_\varepsilon(x) = \int_{\mathbb{R}^3} \varphi_\varepsilon(x - x')\, v(x')\, dx'$$

where $\varphi_\varepsilon(x) = \varepsilon^{-3} \varphi(\varepsilon^{-1} x)$, $\varphi : \mathbb{R}^3 \to \mathbb{R}$ non negative, smooth, compact support in $B(0, 1)$, $\int_{\mathbb{R}^3} \varphi(x)\, dx = 1$.

Exercise 13 *Check that $v_\varepsilon \in C^\infty_{0,per,div}(\mathbb{R}^3, \mathbb{R}^3)$ if $v \in L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$; and that $v_\varepsilon \in \mathcal{D}$ if $v \in H$.*

Exercise 14 *Check that $v_\varepsilon \to v$ in $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$.*

Exercise 15 *If $v \in H^1_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$, check that $v_\varepsilon \to v$ in $H^1_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$.*

These exercises show that $C^{\infty}_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$ is dense in $H^1_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$ and both are dense in $L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$. Similarly, $\mathcal{D}$ is dense in V and both are dense in H.

The embedding $V \subset H$ (and similarly without zero mean condition, and also without zero divergence condition) is continuous: the norm of an element in H is bounded above by the norm in V, up to a constant independent of the element. If we endow V of the norm $\sqrt{|.|_H^2 + \|.\|_V^2}$, this is trivial; if we use the norm $\|.\|_V$, it is Poincaré inequality. More interesting:

Lemma 16 *The embedding $V \subset H$ is compact.*

This means that any bounded set in V is relatively compact in H, or that given a sequence $\{v_n\} \subset V$ with $\sup_n \|v_n\|_V < \infty$, we can extract a subsequence $\{v_{n'}\}$ which converges in the H-topology to an element of H. See appendix B for the proof of a more general result.

2.3 Weak solutions

In a previous section we have seen that heuristic computations, of energy type, on possibly smooth solutions, provide the estimate

$$\sup_{t\in[0,T]} \int_D |u(x,t)|^2 \, dx + \int_0^T \int_D \|\nabla u\|^2 \, dxdt < \infty. \tag{2.1}$$

We have also seen that not much more can be proved even at heuristic level. It is then reasonable to introduce a notion of solution having this very weak level of regularity only.

Assume $u_0 \in L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$, $f \in L^2\left(0,T;L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)\right)$ (this will be generalized below). We say that $u : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$, with property $\sup_{t\in[0,T]} \int_D |u(x,t)|^2 \, dx < \infty$, is weakly continuous in $L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$ if $t \mapsto \int_{[0,L]^3} u(x,t)\cdot\varphi(x)\,dx$ is continuous for every $\varphi \in L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$ (or equivalently, under the uniform in time bound of (2.1), for every $\varphi \in C^{\infty}_{0,per,div}(\mathbb{R}^3,\mathbb{R}^3)$). The condition $u|_{t=0} = u_0$ becomes meaningful in such a case.

Definition 17 *We say that a measurable vector field $u : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ is a periodic weak solution of the Navier-Stokes equations if it satisfies (2.1),*

$\mathtt{div} u = 0$ *in the sense of distributions on* $\mathbb{R}^3$*, it is L-periodic, it is weakly continuous in* $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ *and satisfies* $u|_{t=0} = u_0$*, and the Navier-Stokes equations in the sense of distributions hold, namely*

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \left(\frac{\partial \varphi}{\partial t} + \triangle \varphi \right) dxdt + \int_0^T \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla \varphi)\, dxdt = -\int_0^T \int_{\mathbb{R}^3} f \cdot \varphi dxdt$$

for all $\varphi : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$*, smooth, divergence free and compact support.*

We see that this definition is meaningful even with a lower degree of regularity of u: locally square integrable in (x,t) suffices. A posteriori it seems that there is no point to make this generalization.

It is also easy to see that the previous weak formulation is heuristically a consequence of the Navier-Stokes equations: multiply them by φ and integrate, then integrate by parts both in t and x where it is necessary, and use the identities

$$\int_{\mathbb{R}^3} \varphi \cdot (u \cdot \nabla u)\, dx = -\int_{\mathbb{R}^3} u \cdot (u \cdot \nabla \varphi)\, dx - \int_{\mathbb{R}^3} u \cdot \varphi\, (\mathtt{div} u)\, dx$$

$$\int_{\mathbb{R}^3} \varphi \cdot \nabla p dx = -\int_{\mathbb{R}^3} p \cdot \mathtt{div} \varphi dx$$

plus the fact that $\mathtt{div} u = 0$, $\mathtt{div}\varphi = 0$. Remarkable is that p does not appear anymore in this formulation. It is essential to take divergence free test functions φ. If we, instead, take just smooth compact support fields φ, by the same procedure we get the weak formulation

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \left(\frac{\partial \varphi}{\partial t} + \triangle \varphi \right) dxdt + \int_0^T \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla \varphi)\, dxdt$$
$$= -\int_0^T \int_{\mathbb{R}^3} f \cdot \varphi dxdt - \int_0^T \int_{\mathbb{R}^3} p \cdot \mathtt{div} \varphi dx$$

where p still appears. A reasonable doubt is that, even if we prove the existence of a weak solution u in the sense of the definition, we cannot prove the existence of a pair (u,p) with a locally integrable p satisfying the latter equation, which is a weak formulation closer to the original problem. The reconstruction of p is possible but we postpone it in the next chapter. For many purposes one can ignore the pressure. In this chapter we shall always work without pressure.

2.3.1 Zero average

For technical reasons (essentially the consequences of Poincaré inequality) it is useful to restrict the attention to zero average fields. Let us first argue heuristically. Assume (u, p) is a periodic pair satisfying the Navier-Stokes equations. Introduce

$$m(t) := L^{-3} \int_{[0,L]^3} u(x,t)\, dx$$
$$\psi(t) := L^{-3} \int_{[0,L]^3} f(x,t)\, dx.$$

From the equations we have

$$\frac{dm}{dt} = L^{-3} \int_{[0,L]^3} (-u \cdot \nabla u - \nabla p + \nu \triangle u + f)\, dx = \psi(t)$$

where we have integrated by parts and used periodicity on the boundary and $\mathtt{div} u = 0$. Therefore

$$m(t) = m(0) + \int_0^t \psi(s)\, ds$$

namely $m(t)$ is known from the data. Define

$$v(x,t) := u(x,t) - m(t)$$
$$\widetilde{f}(x,t) := f(x,t) - \psi(t)$$

and check (again heuristically) that (v, p) satisfies

$$\frac{\partial v}{\partial t} + (v + m) \cdot \nabla v + \nabla p = \nu \triangle v + \widetilde{f} \tag{2.2}$$
$$\mathtt{div} v = 0$$
$$v|_{t=0} = v_0.$$

It is just a tedious work to check rigorously that u is a weak solution in the sense of the previous definition if and only if

$$v(x,t) = u(x,t) - m(0) + \int_0^t \psi(s)\, ds$$

is a weak solution of equation (2.2) in the following sense.

Definition 18 *Let $u_0 \in L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in L^2\left(0,T; L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)\right)$ be given. We say that a measurable vector field $v : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ is a periodic weak solution of the zero averaged Navier-Stokes equations if it satisfies (2.1) (with u replaced by v), $\mathtt{div} v = 0$ in the sense of distributions on $\mathbb{R}^3$, it is L-periodic and zero average, it is weakly continuous in H and satisfies $v|_{t=0} = u_0 - m(0)$, and the modified equations (2.2) hold in the sense of distributions, namely*

$$\int_0^T \int_{\mathbb{R}^3} v \cdot \left(\frac{\partial \varphi}{\partial t} + \nu \triangle \varphi \right) dxdt + \int_0^T \int_{\mathbb{R}^3} v \cdot ((v+m) \cdot \nabla \varphi)\, dxdt = -\int_0^T \int_{\mathbb{R}^3} \widetilde{f} \cdot \varphi dxdt$$

for all $\varphi : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$, smooth, divergence free and compact support. Here m and $\widetilde{f}$ are given by the formulae above.

Thus we may restrict ourselves in the sequel to study the modified Navier-Stokes equation (2.2) where the data and the unknown u have zero space-average. Since m in the term $(v+m) \cdot \nabla v$ does not cause any additional difficulty, we develop all the theory for the case $m = 0$, which corresponds to restrict our results to the case of initial conditions and body forces with zero mean. The general case will be given at the end as an exercise, as a subcase of a much more general modified problem of interest in the case of stochastic perturbation.

2.4 Leray-Hopf weak solutions

From now on, as explained in the previous section, we restrict ourselves to the zero mean case.

In the next chapter we shall see that no useful information on the energy balance can be proved for weak solutions. On the contrary, for several reasons it is useful to have a quantitative information on the energy. Hence let us introduce it in the definition of solution; this is meaningful since we shall prove the existence of such solutions.

Definition 19 *Let $u_0 \in H$ and $f \in L^2(0,T;H)$ be given. We say that a measurable vector field $u : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ is a periodic Leray-Hopf*

weak solution of the Navier-Stokes equations if it is a periodic weak solution (namely it is of class

$$u \in L^{\infty}(0,T;H) \cap L^{2}(0,T;V),$$

it is weakly continuous in H, $u|_{t=0} = u_0$, and it satisfies the Navier-Stokes equations in weak form) and in addition it satisfies the energy inequality

$$\begin{aligned}&\frac{1}{2}\int_{[0,L]^d} |u(x,t)|^2\,dx + \nu\int_{t_0}^{t}\int_{[0,L]^d} \|\nabla u\|^2\,dxds \\ &\leq \frac{1}{2}\int_{[0,L]^d} |u(x,t_0)|^2\,dx + \int_{t_0}^{t}\int_{[0,L]^d} u\cdot f dxds\end{aligned}$$

for every $t > 0$ and a.e. $t_0 \in [0,t]$, including $t_0 = 0$.

Usually this definition is given with $t_0 = 0$ only, but in a few cases a free choice of t_0 is useful. The restriction to a.e. $t_0 \in [0,t]$ instead of all $t_0 \in [0,t]$ is essential (it is an open problem whether it is true for all $t_0 \in [0,t]$). The following remark makes explicit one of the possible difficulties.

Remark 20 *Given $t > 0$, let Υ be the set of all $t_0 \in [0,t]$ such that the energy inequality holds. Given $t_0 \in [0,t]$ there exists a sequence $t_0^n \in \Upsilon$ converging to t_0. Clearly $\int_{t_0^n}^{t}\int_{[0,L]^d} \|\nabla u\|^2\,dxds$ converges to $\int_{t_0}^{t}\int_{[0,L]^d} \|\nabla u\|^2\,dxds$ and the same is true for the last integral of the energy inequality. But weak continuity in L^2_{per} only implies*

$$\liminf_{n\to\infty} \frac{1}{2}\int_{[0,L]^d} |u(x,t_0^n)|^2\,dx \geq \frac{1}{2}\int_{[0,L]^d} |u(x,t_0)|^2\,dx.$$

We can deduce only

$$\begin{aligned}&\frac{1}{2}\int_{[0,L]^d} |u(x,t)|^2\,dx + \nu\int_{t_0}^{t}\int_{[0,L]^d} \|\nabla u\|^2\,dxds \\ &\leq \liminf_{n\to\infty} \frac{1}{2}\int_{[0,L]^d} |u(x,t_0^n)|^2\,dx + \int_{t_0}^{t}\int_{[0,L]^d} u\cdot f dxds\end{aligned}$$

but we cannot say that the energy inequality holds at t_0.

Remark 21 *On the contrary, it is equivalent to ask the energy inequality for every $t > 0$ as we have done or only for a.e. $t > 0$. Indeed, If it holds for a sequence t_n converging to t, then the double integral pass to the limit and we get*

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{2}\int_{[0,L]^d} |u(x,t_n)|^2\,dx + \nu\int_{t_0}^{t}\int_{[0,L]^d} \|\nabla u\|^2\,dxds \\ \leq \frac{1}{2}\int_{[0,L]^d} |u(x,t_0)|^2\,dx + \int_{t_0}^{t}\int_{[0,L]^d} u\cdot f dxds\end{aligned}$$

which implies the inequality at time t.

Replacing $\int_{[0,L]^d} u(t,x)\cdot f(t,x)\,dx$ by

$$\langle f(t), u(t,.)\rangle_{V',V}$$

we may assume only

$$f \in L^2(0,T;V').$$

For notations and a discussion about V' see appendix A.

Theorem 22 *If $u_0 \in H$ and $f \in L^2(0,T;V')$ there exists a periodic Leray-Hopf weak solution of the Navier-Stokes equations. Moreover, $u \in W^{1,4/3}(0,T;V')$.*

This is the main theorem of this chapter. The next sections will be devoted to its proof.

2.5 Compactness theorems

The strategy we adopt to prove existence of Leray-Hopf weak solutions is the so called compactness method (see [7] for a clear classification of some approaches to nonlinear problems). Since we deal with functions of time and space and the expected regularity in the two variable is different, we need special theorems that put together a form of compactness in time with one in space. To explain better, although spaces of differentiable functions (in the Sobolev sense) have good compactness properties in Lebesgue spaces (see for instance exercise **??** above), over bounded interval of time or bounded sets in space, we do not have good bounds for space-time derivatives simultaneously, so we cannot just apply such compactness results jointly in space-time. What

we have are good estimates of first space derivatives, but only L^2 in time (see (2.1)), and we shall prove some estimate of time derivatives but in very weak topologies in space. The main theorem below states that, in a proper sense, it is sufficient to have compactness in space and in time separately.

Ascoli-Arzelà theorem in a Banach space B states that a family G of functions $f : [0,T] \to B$ such that

- for every $t \in [0,T]$, the set $\{f(t) ; f \in G\}$ is relatively compact
- the elements of G are uniformly equicontinuous

then G is a relatively compact set of $C([0,T] ; B)$.

We recall that relatively compact means that the closure is compact, or in other words that from any sequence it is possible to extract a converging subsequence (but the limit may be in the relatively compact set). A set is relatively compact if and only if it is totally bounded, namely for every $\varepsilon > 0$ it may be covered by a finite number of balls of radius smaller than ε. A family G is uniformly equicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ if $|t - s| < \delta$, independently of $t, s \in [0,T]$ and $f \in G$.

Exercise 23 *A good proof of Ascoli-Arzelà theorem can be found even on Wikipedia (Internet). Try to reconstruct it starting from the following idea. Let $\Upsilon \subset [0,T]$ be a dense countable set. The first step is to find a sequence $\{f_n\}$ which converges pointwise on Υ. The second step is to prove that $\{f_n\}$ converges uniformly.*

Using Ascoli-Arzelà theorem we can prove a compactness result in L^p topologies. We say that the embedding $B_0 \subset B$ of two Banach spaces is continuous if

$$\|x\|_B \leq C \|x\|_{B_0} \quad \text{for every } x \in B_0$$

for some constant $C > 0$, and we say that the embedding is compact if in addition any bounded set in B_0 is relatively compact in B.

Lemma 24 *Let $p > 1$ and let $B_0 \subset B$ be two Banach spaces with compact embedding. A family G of functions $f : [0,T] \to B$ such that*

- $$\sup_{f \in G} \int_0^T \|f(s)\|_{B_0}^p \, ds < \infty$$

- *there exist* $\theta, C > 0$ *such that*

$$\int_0^{T-h} \|f(t+h) - f(t)\|_B^p \, dt \leq C |h|^\theta$$

is relatively compact in $L^p(0,T';B)$ *for every* $T' \in (0,T)$ *and in* $L^{p'}(0,T;B)$ *for every* $p' \in (1,p)$.

Proof. Step 1. A number $T' \in (0,T)$ is given throughout the proof of steps 1,2,3. Set also $\varepsilon_0 = T - T'$.

Let $\varepsilon \in (0, \varepsilon_0]$ be given. Let us show that the family G_ε of functions defined as

$$f_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(s) \, ds, \quad t \in [0,T']$$

when f varies in G, satisfies the assumptions of Ascoli-Arzelà theorem, hence it is relatively compact in $C([0,T'];B)$. Given $t \in [0,T']$, let us consider the set in B of all values $f_\varepsilon(t)$, $f \in G$. We have

$$\|f_\varepsilon(t)\|_{B_0} \leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \|f(s)\|_{B_0} \, ds \leq C \left(\int_t^{t+\varepsilon} \|f(s)\|_{B_0}^p \, ds \right)^{1/p} \leq C$$

where C is a generic constant (depending only on ε and G). Since bounded sets in B_0 are relatively compact in B, the first condition of Ascoli-Arzelà theorem is proved. For the uniform equicontinuity in B let us estimate $\|f_\varepsilon(t) - f_\varepsilon(\tau)\|_B$ when $0 \leq t \leq \tau \leq T'$: since

$$f_\varepsilon(\tau) = \frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} f(s) \, ds = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(s + (\tau - t)) \, ds$$

we have

$$f_\varepsilon(\tau) - f_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (f(s + (\tau - t)) - f(s)) \, ds$$

$$\|f_\varepsilon(\tau) - f_\varepsilon(t)\|_B \leq C \left(\int_t^{t+\varepsilon} \|f(s + (\tau - t)) - f(s)\|_B^p \, ds \right)^{1/p} \leq C |\tau - t|^{\theta/p}$$

by the second assumption of the lemma, so the proof of uniform equicontinuity in B is complete. By Ascoli-Arzelà theorem, G_ε is relatively compact in $C([0,T'];B)$.

Step 2. The functions f_ε, for $\varepsilon \in (0, \varepsilon_0]$, are uniformly close to f in the family G, in the $L^p(0,T';B)$-topology:

$$\int_0^{T'} \|f_\varepsilon(t) - f(t)\|_B^p \, dt = \int_0^{T'} \left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (f(s) - f(t)) \, ds \right\|_B^p dt$$
$$\leq \int_0^{T'} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \|f(s) - f(t)\|_B^p \, ds dt$$
$$= \int_0^{T'} \frac{1}{\varepsilon} \int_0^{\varepsilon} \|f(t+h) - f(t)\|_B^p \, dh dt$$

$$= \frac{1}{\varepsilon} \int_0^{\varepsilon} \left(\int_0^{T'} \|f(t+h) - f(t)\|_B^p \, dt \right) dh$$
$$\leq \frac{1}{\varepsilon} \int_0^{\varepsilon} Ch^\theta dh \leq C\varepsilon^\theta.$$

Step 3. Let $\{f^n\} \subset G$ be any sequence. We prove it has a subsequence that converges in $L^p(0,T';B)$; this means that G is relatively compact in $L^p(0,T';B)$. It is sufficient to prove that $\{f^n\}$ is a Cauchy sequence in $L^p(0,T';B)$: given $\widetilde{\varepsilon}$, we need to find n_0 such that

$$\int_0^{T'} \|f^n(t) - f^m(t)\|_B^p \, dt \leq \widetilde{\varepsilon} \text{ for all } n, m \geq n_0.$$

For all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\int_0^{T'} \|f^n(t) - f^m(t)\|_B^p \, dt$$

$$\leq \int_0^{T'} \|f^n(t) - f_\varepsilon^n(t)\|_B^p \, dt + \int_0^{T'} \|f_\varepsilon^n(t) - f_\varepsilon^m(t)\|_B^p \, dt$$
$$+ \int_0^{T'} \|f_\varepsilon^m(t) - f^m(t)\|_B^p \, dt$$

and by step 2

$$\leq 2C\varepsilon^\theta + \int_0^{T'} \|f_\varepsilon^n(t) - f_\varepsilon^m(t)\|_B^p \, dt.$$

Choose ε such that $2C\varepsilon^{\theta} \leq \frac{\widetilde{\varepsilon}}{2}$. For that value of ε, apply the compactness of the sequence $\{f_{\varepsilon}^{n}\}$ in $C\left([0,T'];B\right)$ to deduce that there exists n_0 such that

$$\int_0^{T'} \left\| f_{\varepsilon}^{n}(t) - f_{\varepsilon}^{m}(t) \right\|_B^p dt \leq \frac{\widetilde{\varepsilon}}{2}$$

for all $n, m \geq n_0$. Thus

$$\int_0^{T'} \left\| f^{n}(t) - f^{m}(t) \right\|_B^p dt \leq \widetilde{\varepsilon}$$

for all $n, m \geq n_0$. The proof of relative compactness in $L^p(0,T';B)$ is complete.

Step 4. Given $\varepsilon_1 \in (0, p-1)$, let us finally prove the relative compactness in $L^{p'}(0,T;B)$ for a given $p' \in (1,p)$. Choose any $\varepsilon_0 > 0$ and call G' the family of functions $f : [0, T+\varepsilon_0] \to B$ that are equal to an element of G on $[0,T]$ and equal to zero on $(T, T+\varepsilon_0]$. The family G' satisfies on $[0, T+\varepsilon_0]$ the assumptions of the theorem, but with p' in place of p, and a different value of θ. Indeed, the first assumption is obvious, while the second one comes from

$$\begin{aligned}
&\int_0^{T+\varepsilon_0-h} \left\| f(t+h) - f(t) \right\|_B^{p'} dt \\
&= \int_0^{T-h} \left\| f(t+h) - f(t) \right\|_B^{p'} dt + \int_{T-h}^{T+\varepsilon_0-h} \left\| f(t+h) - f(t) \right\|_B^{p'} dt \\
&\leq C|h|^{\theta} + 2\int_{T-h}^{T} \left\| f(t) \right\|_B^{p'} dt \\
&\leq C|h|^{\theta} + 2\left(\int_{T-h}^{T} \left\| f(t) \right\|_B^{p} dt \right)^{p'/p} |h|^{\alpha}
\end{aligned}$$

for some $\alpha > 0$. Thus we can apply the first part of the lemma (the one proved in steps 1,2,3) and get the relative compactness in $L^{p'}(0,T;B)$. The proof is complete. ■

We can now state the main result of interest for us.

Theorem 25 *Let $p_0, p_1 > 1$ and let $B_0 \subset B \subset B_1$ be three Banach spaces, the embedding $B_0 \subset B$ being compact, the embedding $B_0 \subset B$ being continuous. Assume that G is a family of functions $f : [0,T] \to B_1$ such that*

- $$\sup_{f\in G}\int_0^T \|f(s)\|_{B_0}^{p_0}\,ds<\infty$$
- *there exist* $\theta, C>0$ *such that*

$$\int_0^{T-h} \|f(t+h)-f(t)\|_{B_1}^{p_1}\,dt\le C\,|h|^{\theta}.$$

Then G *is relatively compact in* $L^p(0,T;B)$ *for every* $p\in(1,p_0)$.

Proof. Step 1. Let us prove the following important general claim. If $B_0\subset B\subset B_1$ are like in the assumptions of the theorem, then given $\varepsilon>0$ there is a constant $C(\varepsilon)>0$ such that

$$\|x\|_B\le \varepsilon\|x\|_{B_0}+C(\varepsilon)\|x\|_{B_1}\quad \text{for every } x\in B_0.$$

Let us prove this claim by contradiction. Thus assume that there is $\varepsilon_0>0$ such that for every n there is $x_n\in B_0$ with

$$\|x_n\|_B\ge \varepsilon_0\|x_n\|_{B_0}+n\|x_n\|_{B_1}.$$

Set $y_n=\frac{x_n}{\|x_n\|_{X_0}}$, so we have

$$\|y_n\|_B\ge \varepsilon_0+n\|y_n\|_{B_1}.$$

Since $\|y_n\|_{B_0}=1$, the sequence $\{y_n\}$ is bounded in B_0, hence it converges in B to some $y\in B$. By continuous embedding of B into B_1, it converges in the topology of B_1 too. Since $n\|y_n\|_{B_1}\le 1-\varepsilon_0$, necessarily $y=0$. But this is in contradiction with $\|y_n\|_B\ge\varepsilon_0$. The claim is proved.

Step 2. Let us use step 1 to prove a second important general claim. If a set of functions G is bounded in $L^{p_0}(0,T;B_0)$ and relatively compact in $L^{p_1}(0,T;B_1)$ then it is relatively compact in $L^p(0,T;B)$, both for any $p\in(1,p_0)$. Here $B_0\subset B\subset B_1$ must satisfy the assumptions of the theorem.

Denote $\min(p_0,p_1)$ by p^*. It is a simple exercise to check that the assumptions of the claim are satisfied with p^* in place of p_0 and p_1. Let us first prove the result with p^* in place of any $p\in(1,p_0)$.

Let $\{u_n\}$ be a sequence in G that, by the assumptions, we may assume convergent in $L^{p_1}(0,T;B_1)$, hence a Cauchy sequence. From step 1, for every $\varepsilon'>0$ there is $C(\varepsilon')>0$ such that

$$\|u_n-u_m\|_{L^{p^*}(0,T;B)}\le \varepsilon'\|u_n-u_m\|_{L^{p^*}(0,T;B_0)}+C(\varepsilon')\|u_n-u_m\|_{L^{p^*}(0,T;B_1)}.$$

Given $\varepsilon > 0$, let $\varepsilon' > 0$ be such that

$$2\varepsilon' \left\| u_n \right\|_{L^{p^*}(0,T;B_0)} \leq \frac{\varepsilon}{2}$$

for every n (it exists by assumption). Let n_0 be such that

$$C\left(\varepsilon'\right) \left\| u_n - u_m \right\|_{L^{p^*}(0,T;B_1)} \leq \frac{\varepsilon}{2}$$

for every $n, m \geq n_0$. Then

$$\left\| u_n - u_m \right\|_{L^{p^*}(0,T;B)} \leq \varepsilon$$

for every $n, m \geq n_0$. This proves the claim for p^*.

Take now $p \in (1, p_0)$. Since $\{u_n\}$ is bounded in $L^{p_0}(0,T;B_0)$ then it is bounded in $L^{p_0}(0,T;B)$. Passing to a subsequence, we know that $\{u_n\}$ converges in $L^{p^*}(0,T;B)$ and thus, for a further subsequence $\{u_{n_k}\}$, we can say that $\{u_{n_k}\}$ converges almost surely in B. Summarizing, $\{u_{n_k}\}$ is bounded in $L^{p_0}(0,T;B)$ and converges almost surely in B. By Vitali theorem, it converges in $L^p(0,T;B)$. The proof is complete.

Step 3. The assumptions, just using the previous lemma, imply relative compactness in $L^{p_1'}(0,T;B_1)$ for every $p_1' \in (1, p_1)$. In addition G is bounded in $L^{p_0}(0,T;B_0)$ (by assumption). Hence by step 2 it is relatively compact in $L^p(0,T;B)$ for every $p \in (1, p_0)$. The proof is complete. ■

For $\alpha \in (0,1)$ and $p \geq 1$, denote by $W^{\alpha,p}(0,T;B)$ the fractional order Sobolev space of all $f \in L^p(0,T;B)$ such that

$$\int_0^T \int_0^T \frac{\left\| f(t) - f(s) \right\|_B^p}{|t-s|^{1+\alpha p}} ds dt < \infty.$$

Corollary 26 *Let $p_0, p_1 > 1$, $\alpha \in (0,1)$ and let $B_0 \subset B \subset B_1$ be three Banach spaces, the embedding $B_0 \subset B$ being compact, the embedding $B_0 \subset B$ being continuous. If a family G of functions $f : [0,T] \to B_1$ is bounded in $L^{p_0}(0,T;B_0)$ and in $W^{\alpha,p_1}(0,T;B_1)$ then it is relatively compact in $L^p(0,T;B)$ for every $p \in (1, p_0)$.*

Proof. Let $r \in (1, p_1)$. It is sufficient to notice that (take $h > 0$)

$$\begin{aligned} \left\| f(t+h) - f(t) \right\|_{B_1}^r \leq \frac{C_p}{h} \int_t^{t+h} \left\| f(t+h) - f(s) \right\|_{B_1}^r ds \\ + \frac{C_p}{h} \int_t^{t+h} \left\| f(s) - f(t) \right\|_{B_1}^r ds \end{aligned}$$

$$\int_0^{T-h} \|f(t+h)-f(t)\|_{B_1}^r \, dt \le \frac{C_p}{h} \int_0^{T-h} \int_t^{t+h} \|f(t+h)-f(s)\|_{B_1}^r \, dsdt$$
$$+\frac{C_p}{h} \int_0^{T-h} \int_t^{t+h} \|f(s)-f(t)\|_{B_1}^r \, dsdt$$

and (we handle only the second integral, the first one being similar)

$$\int_0^{T-h} \int_t^{t+h} \|f(s)-f(t)\|_{B_1}^r \, dsdt$$
$$= \int_0^{T-h} \int_t^{t+h} \frac{\|f(s)-f(t)\|_{B_1}^r}{|t-s|^{r(1+\alpha p_1)/p_1}} |t-s|^{r(1+\alpha p_1)/p_1} \, dsdt$$
$$\le \left(\int_0^{T-h} \int_t^{t+h} \frac{\|f(s)-f(t)\|_{B_1}^{p_1}}{|t-s|^{1+\alpha p_1}} dsdt \right)^{r/p_1}$$
$$\cdot \left(\int_0^{T-h} \int_t^{t+h} |t-s|^{\frac{r(1+\alpha p_1)}{p_1-r}} \, dsdt \right)^{(p_1-r)/p_1}$$
$$\le Ch^{\left(\frac{r(1+\alpha p_1)}{p_1-r}+1\right)\frac{p_1-r}{p_1}} = Ch^{\frac{r(1+\alpha p_1)}{p_1}+\frac{p_1-r}{p_1}}.$$

Hence

$$\int_0^{T-h} \|f(t+h)-f(t)\|_{B_1}^r \, dt \le Ch^{\frac{r(1+\alpha p_1)}{p_1}+\frac{p_1-r}{p_1}-1}.$$

This implies the validity of the second assumption of the theorem, if we choose r sufficiently close to p_1. The proof is complete. ■

Exercise 27 *If $\alpha p > 1$, the space $W^{\alpha,p}(0,T;B)$ is contained in the space $C^{\beta}([0,T];B)$ of β-Hölder continuous functions on $[0,T]$ with values in B, for every $\beta \in (0, \alpha p - 1)$. Giving this fact for granted, use it to give an easier proof of the previous corollary.*

Exercise 28 *Let $p_0, p_1 > 1$ and let $B_0 \subset B \subset B_1$ be three Banach spaces, the embedding $B_0 \subset B$ being compact, the embedding $B_0 \subset B$ being continuous. If a family G of functions $f : [0,T] \to B_1$ is bounded in $L^{p_0}(0,T;B_0)$ and in $W^{1,p_1}(0,T;B_1)$ then it is relatively compact in $L^p(0,T;B)$ for every $p \in (1,p_0)$. Prove this simpler claim.*

2.6 Existence of Leray-Hopf weak solutions

In this section we want to prove theorem 22 of existence of Leray-Hopf weak solutions

As we have already said, the idea we are going to develop is the so called *method of compactness*: 1) introduce a sequence of approximating problems which are solvable; 2) prove uniform estimates on their solution; 3) deduce, from compactness theorems, that a subsequence of these solutions converge in proper topologies to a field u; 4) pass to the limit and prove that u is a weak Leray-Hopf solution of the Navier-Stokes equations.

Two efficient ways to perform point 1 are: 1a) finite dimensional approximation (Galerkin, or Faedo-Riesz-Galerkin method); 1b) Leray approximation, which consists in smoothing the driving field in the inertial term, $u_\varepsilon \cdot \nabla u$. Let us describe 1a in detail and postpone 1b to the next chapter. The method 1a has also the advantage to look similar to numerical methods.

It is useful to have one more space. We denote by $D(A)$ (the reason for this notation will be clear only later on) the space of all $v \in V$ such that second derivatives in the sense of distributions are square integrable:

$$\partial_i \partial_j u_k \in L^2\left([0,L]^3\right)$$

for all $i,j,k = 1,2,3$. We have $D(A) \subset V \subset H$.

2.6.1 Galerkin approximations

Collect some informations from appendix B to solve the following exercise.

Exercise 29 *Find a sequence $\lambda_n \to \infty$ of non negative real numbers and a sequence of vector fields $\{h_n\} \subset D(A)$ such that $\{h_n\}$ is a complete orthonormal system in H and h_n is eigenvector of the Stokes operator with eigenvalue λ_n, in the sense that*

$$-\nu \triangle h_n = \lambda h_n \text{ in } \mathbb{R}^3.$$

The fields h_n are more concretely parametrized by wave number $k \in \mathbb{Z}^d \diagdown \{0\}$ and an additional parameter $j = 1, ..., d-1$, but this parametrization obscures the simplicity of the method we are going to describe.

Let us introduce a number of objects:

- H_n is the sub-space of H spanned by $h_1, ..., h_n$; it is isomorphic to $\mathbb{R}^n$

- $\pi_n : H \to H$ is the orthogonal projector on H_n, defined as

$$\pi_n v = \sum_{j=1}^{n} \langle v, h_j \rangle_H h_j$$

- $A_n : H_n \to H_n$ is the linear operator defined as

$$A_n v = \nu \sum_{j=1}^{n} \lambda_j \langle v, h_j \rangle_H h_j$$

- $B_n : H_n \times H_n \to H_n$ is the linear operator defined as

$$B_n(u, v) = \sum_{j=1}^{n} \langle (u \cdot \nabla v), h_j \rangle_H h_j$$

- f_n and $u_{0,n}$ are given by

$$f_n(t) = \sum_{j=1}^{n} \langle f(t), h_j \rangle_{V',V} h_j, \quad u_{0,n} = \sum_{j=1}^{n} \langle u_0, h_j \rangle_H h_j.$$

Exercise 30 *Useful basic properties of these objects are* $(u, v, z \in H_n)$*:*
1) $\pi_n \circ \pi_n = \pi_n$ *and*

$$\langle \pi_n u, v \rangle_H = \langle u, \pi_n v \rangle_H = \langle \pi_n u, \pi_n v \rangle_H$$

2)

$$\langle A_n u, v \rangle_H = \langle u, A_n v \rangle_H$$

3)

$$\langle A_n u, u \rangle_H \geq \nu \lambda_1 |u|_H^2$$

4)

$$\langle A_n u, u \rangle_H = \nu \|u\|_V^2$$

hence also $\langle A_n u, u \rangle_H \geq \nu \lambda |u|_H^2$ *where* λ *is Poincaré constant*
5)

$$\langle B_n(u, v), z \rangle_H = -\langle B_n(u, z), v \rangle_H$$

6)

$$\langle B_n(u,v), v\rangle_H = 0$$

7)

$$|\pi_n v|_H^2 \leq |v|_H^2, \quad \|\pi_n v\|_V^2 \leq \|v\|_V^2, \quad \|\pi_n v\|_{V'}^2 \leq \|v\|_{V'}^2$$

where we understand that π_n is extended to a linear operator from V' to H_n defined as $\pi_n v = \sum_{j=1}^n \langle v, h_j\rangle_{V',V} h_j$, and H_n is considered as a subspace of V' following the identifications described in appendix A

8)

$$\langle f_n(t), v\rangle_H \leq \|f_n(t)\|_{V'} \|v\|_V.$$

See also the next remark to understand better some of these properties.

Remark 31 *Properties 2, 4, 5, 6 are the projected analog of*

$$\int_{[0,L]^3} \triangle u \cdot v dx = -\int_{[0,L]^3} \nabla u \cdot \nabla v dx = \int_{[0,L]^3} u \cdot \triangle v dx$$

$$\int_{[0,L]^3} (u\cdot\nabla v)\cdot\varphi dx = -\int_{[0,L]^3} (u\cdot\nabla\varphi)\cdot v dx - \int_{[0,L]^3} (v\cdot\varphi)\,\mathtt{div} u dx$$
$$= -\int_{[0,L]^3} (u\cdot\nabla\varphi)\cdot v dx$$

$$\int_{[0,L]^3} (u\cdot\nabla v)\cdot v dx = \frac{1}{2}\int_{[0,L]^3} u\cdot\nabla |v|^2 dx$$
$$= -\frac{1}{2}\int_{[0,L]^3} |v|^2 \,\mathtt{div} u dx = 0$$

where all boundary terms disappear because of periodicity. See also the next remark.

With these notations, consider the following finite dimensional differential equation in the n-dimensional Euclidean space H_n:

$$\frac{du_n}{dt} + A_n u_n + B_n(u_n, u_n) = f_n, \quad t\in[0,T]$$
$$u_n(0) = u_{0,n}.$$

Remark 32 *The definition of A_n and B_n becomes clear if we introduce the following operators A and B: $A : D(A) \subset H \to H$ is defined as*

$$Av = \nu \triangle v, \quad v \in D(A)$$

and $B : D(A) \times D(A) \to H$ is defined in the following way: for $u, v \in D(A)$, $B(u, v)$ is the unique element of H such that

$$\langle B(u, v), \varphi \rangle_H = \int_{[0,L]^3} (u \cdot \nabla v) \cdot \varphi dx$$

for every $\varphi \in H$ (see the next exercise). Then

$$A_n v = \pi_n A v, \quad B_n(v, v) = \pi_n B(v, v).$$

Thus the finite dimensional equation is a sort of π_n-projected version of the abstract equation

$$\frac{du}{dt} + Au + B(u, u) = f, \quad t \in [0, T].$$

We could show that this equation, interpreted in more generalized way (namely interpreting the time derivative in the distributional sense and extending A from V to V' and B from $V \times V$ to V') is equivalent to the weak formulation of the Navier-Stokes equations.

Exercise 33 *Prove that $u, v \in D(A)$, $\varphi \in H$ implies*

$$\int_{[0,L]^3} |u| \, |\nabla v| \, |\varphi| \, dx < \infty.$$

Use the fact that all elements of $D(A)$ are continuous functions (this will be proved somewhere else). Notice also that a similar remark is needed in the definition of B_n.

The equation for u_n has a unique global solution. Indeed, it satisfies the assumptions of the following well known theorem. In $\mathbb{R}^d$, consider the Cauchy problem

$$\frac{du(t)}{dt} = F(t, u(t)) + g(t), \quad u(0) = u_0.$$

We say that $u \in C\left([0, T]; \mathbb{R}^d\right)$ is a solution if

$$u(t) = u_0 + \int_0^t F(s, u(s)) \, ds + \int_0^t g(s) \, ds \text{ for } t \in [0, T].$$

Theorem 34 *Assume g is square integrable, $F(t,x)$ is continuous, locally Lipschitz continuous in x uniformly in t, and*

$$\langle F(t,x), x\rangle \leq C\left(|x|^2+1\right) \text{ for all } x\in\mathbb{R}^d$$

for some constant $C>0$. Then there exists a unique solution $u\in C\left([0,T];\mathbb{R}^d\right)$. The solution is also of class $W^{1,2}\left(0,T;\mathbb{R}^d\right)$.

We omit the proof also because we shall write details of a more general statement in a later chapter in the stochastic case. Only, let us say that local existence and uniqueness comes from the locally Lipschitz assumption, while non explosion (global solution) is a consequence of the assumed inequality on F and the energy-type computation

$$\begin{aligned}\frac{d|u|^2}{dt} &= 2\langle F(t,u),u\rangle + \langle g,u\rangle\\ &\leq C\left(|u|^2+1\right)+|g|^2+|u|^2\end{aligned}$$

to which application of Gronwall lemma follows.

The Galerkin system above satisfies the conditions of the theorem. because $\langle B_n(v,v),v\rangle_H=0$ and $\langle A_nv,v\rangle_H\geq 0$.

To summarize, we have introduced the Galerkin system, which is well posed.

2.6.2 Uniform estimates

We know that the solution u_n of Galerkin system is of class $W^{1,2}(0,T;H_n)$ and $C([0,T];H_n)$. With some care one can prove that $|u_n|^2\in W^{1,2}(0,T;\mathbb{R})$ and

$$\frac{1}{2}\frac{d|u_n|_H^2}{dt}=\left\langle u_n,\frac{du_n}{dt}\right\rangle_H.$$

If the reader does not like to use this fact, it is sufficient to approximate f_n further by means of continuous functions of time, have $u_n\in C^1([0,T];H_n)$ and use ordinary calculus.

From Galerkin system we deduce

$$\frac{1}{2}\frac{d|u_n|_H^2}{dt}+\langle A_nu_n,u_n\rangle_H=\langle f_n,u_n\rangle_H$$

since $\langle B_n(u_n, u_n), u_n\rangle_H = 0$. Moreover, $\langle A_n u_n, u_n\rangle_H = \nu \|u_n\|_V^2$, hence

$$\begin{aligned}|u_n(t)|_H^2 + \nu \int_0^t \|u_n(s)\|_V^2 \, ds &= |u_n(0)|_H^2 + \int_0^t \langle f_n(s), u_n(s)\rangle_H \, ds \\ &\leq |u_n(0)|_H^2 + \int_0^t \|f_n(s)\|_{V'} \|u_n(s)\|_V \, ds \\ &\leq |u_n(0)|_H^2 + \int_0^t \|f_n(s)\|_{V'} \|u_n(s)\|_V \, ds.\end{aligned}$$

Recall that

$$ab = 2\left(\sqrt{\varepsilon}a\right)\left(\frac{1}{2\sqrt{\varepsilon}}b\right) \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$$

hence, for $\varepsilon = \frac{\nu}{2}$,

$$\begin{aligned}&\int_0^t \|f_n(s)\|_{V'} \|u_n(s)\|_V \, ds \\ &\leq \frac{\nu}{2}\int_0^t \|u_n(s)\|_V^2 \, ds + \frac{1}{2\nu}\int_0^t \|f_n(s)\|_{V'}^2 \, ds.\end{aligned}$$

Summarizing,

$$|u_n(t)|_H^2 + \frac{\nu}{2}\int_0^t \|u_n(s)\|_V^2 \, ds \leq |u_n(0)|_H^2 + \frac{1}{2\nu}\int_0^t \|f_n(s)\|_{V'}^2 \, ds.$$

We finally obtain

$$\sup_{t\in[0,T]} |u_n(t)|_H^2 + \frac{\nu}{2}\int_0^T \|u_n(s)\|_V^2 \, ds \leq |u(0)|_H^2 + \frac{1}{2\nu}\int_0^t \|f(s)\|_{V'}^2 \, ds. \quad (2.3)$$

This is the analog of estimate (2.1).

In view of the use of compactness theorems, we need sone compactness in time too. This is provided by the previous estimates and the equation itself. This is perhaps the only technical point of the proof.

Lemma 35

$$\int_0^T \left\|\frac{du_n(t)}{dt}\right\|_{V'}^{4/3} dt \leq C$$

where C depends only on $|u(0)|_H^2$, $\int_0^t \|f(s)\|_{V'}^2 \, ds$, ν, L.

Proof. Consider H_n as a subset of V' (h_j are the elements of V' defined by $h_j(\varphi) = \langle h_j, \varphi \rangle_H$ for every $\varphi \in V$). Then, from the Galerkin system,

$$\left\| \frac{du_n}{dt} \right\|_{V'} \leq \|A_n u_n\|_{V'} + \|B_n(u_n, u_n)\|_{V'} + \|f_n\|_{V'}$$
$$= I_1^{(n)} + I_2^{(n)} + I_3^{(n)}.$$

We know that $\|f_n(t)\|_{V'} \leq \|f(t)\|_{V'}$, so $I_3^{(n)}$ is equibounded in $L^2(0,T)$. There exists a constant $C > 0$ such that

$$\|A_n u_n\|_{V'} \leq C \|u_n\|_V$$

since

$$\|A_n u_n\|_{V'} = \sup_{\|\varphi\|_V \leq 1} |A_n u_n(\varphi)|$$

where, using several facts discussed above,

$$A_n u_n(\varphi) = \langle A_n u_n, \varphi \rangle_H = \langle \pi_n A u_n, \varphi \rangle_H = \langle A u_n, \pi_n \varphi \rangle_H$$
$$= -\nu \int_{[0,L]^3} \triangle u_n \cdot \pi_n \varphi dx = \int_{[0,L]^3} \nabla u_n \cdot \nabla \pi_n \varphi dx$$

(boundary terms disappear because of periodicity), so that

$$|A_n u_n(\varphi)| \leq \nu \|u_n\|_V \|\pi_n \varphi\|_V \leq \nu \|u_n\|_V \|\varphi\|_V.$$

Therefore $I_1^{(n)}$ is equibounded in $L^2(0,T)$, from (2.3).

As to $\|B_n(u_n, u_n)\|_{V'}$ we have

$$\|B_n(u_n, u_n)\|_{V'} = \sup_{\|\varphi\|_V \leq 1} |\langle B_n(u_n, u_n), \varphi \rangle_H|$$

$$|\langle B_n(u_n, u_n), \varphi \rangle_H| = \left| \int_{[0,L]^3} (u_n \cdot \nabla u_n) \cdot \pi_n \varphi dx \right|$$
$$\leq \int_{[0,L]^3} |u_n| |\nabla u_n| |\pi_n \varphi| dx$$
$$\leq C \|u_n\|_V \left(\int_{[0,L]^3} |\pi_n \varphi|^6 dx \right)^{1/6} \left(\int_{[0,L]^3} |u_n|^3 dx \right)^{1/3}$$

$$\leq C \|u_n\|_V |u_n|_H^{1/2} \|u_n\|_V^{1/2} \|\varphi\|_V$$
$$= C |u_n|_H^{1/2} \|u_n\|_V^{3/2} \|\varphi\|_V$$

hence

$$\|B_n(u_n, u_n)\|_{V'} \leq C |u_n|_H^{1/2} \|u_n\|_V^{3/2}.$$

Here C denotes a generic constant depending only on L. We have used Hölder inequality, the Sobolev embedding

$$V \subset L^6\left([0,L]^3\right)$$

and the multiplicative inequality

$$\|v\|_{L^3\left([0,L]^3\right)} \leq C_L |v|_H^{1/2} \|v\|_V^{1/2}, \quad v \in V$$

discussed in the appendix to Chapter 1. From from (2.3) we see that $I_2^{(n)}$ is equibounded in $L^{4/3}(0,T)$. The proof is complete. ■

Remark 36 *The poor estimate of $I_2^{(n)}$ in $L^{4/3}(0,T)$ instead of $L^2(0,T)$ is one of the very many (often equivalent) origins of the troubles and open problems of the 3D Navier-Stokes equations.*

Exercise 37 *Use the integral form of the equation, namely*

$$u_n(t) + \int_\tau^t A_n u_n(s)\, ds + \int_\tau^t B_n(u_n, u_n)\, ds = u_n(\tau) + \int_\tau^t f_n(s)\, ds$$

to check the integral criterium

$$\int_0^{T-h} \|u_n(t+h) - u_n(t)\|_{V'}^p\, dt \leq C |h|^\theta$$

directly, without using the derivative of u_n. This is instructive in view of the stochastic case, when the processes are not differentiable in time.

2.6.3 Compactness and passage to the limit

The previous estimates tell us that $\{u_n\}$ is bounded in $L^2(0,T;V)$ and in $W^{1,4/3}(0,T;V')$. Thus it has a subsequence $\{u_{n_k}\}$ which converges to some function u in $L^2(0,T;H)$. Here we have used the fact that the embedding $V \subset H$ is compact (exercise **??**), the embedding (after proper identifications) $H \subset V'$ is continuous, and we have used exercise 28.

We may also assume, passing to a further subsequence, that:

- $\{u_{n_k}\}$ converges to u also almost surely in time, in the topology of H
- $\{u_{n_k}\}$ weakly converges to some v in $L^2(0,T;V)$:

$$\int_0^T \langle \varphi(t), u_{n_k}(t)\rangle_{V',V}\, dt \to \int_0^T \langle \varphi(t), v(t)\rangle_{V',V}\, dt$$

  for every $\varphi \in L^2(0,T;V')$
- $\{u_{n_k}\}$ converges weak star to some w in $L^\infty(0,T;H)$:

$$\int_0^T \langle \varphi(t), u_{n_k}(t)\rangle_H\, dt \to \int_0^T \langle \varphi(t), v(t)\rangle_H\, dt$$

  for every $\varphi \in L^1(0,T;H)$
- $\left\{\frac{du_{n_k}}{dt}\right\}$ weakly converges to some z in $L^{4/3}(0,T;V')$:

$$\int_0^T \left\langle \frac{du_{n_k}(t)}{dt}, \varphi(t)\right\rangle_{V',V} dt \to \int_0^T \langle z(t), \varphi(t)\rangle_{V',V}\, dt$$

  for every $\varphi \in L^4(0,T;V)$.

These are known weak compactness results in functional analysis for which we address the reader to [8].

Exercise 38 *Show that $v = w = u$, $u \in W^{1,4/3}(0,T;V')$ and $\frac{du}{dt} = z$.*

Proof. Hint: check that both weak convergence in $L^2(0,T;V)$, weak star convergence in $L^\infty(0,T;H)$ and strong convergence in $L^2(0,T;H)$ imply weak convergence in $L^2(0,T;H)$. Check also that weak convergence of functions and their derivatives in $L^{4/3}(0,T;V')$ imply that the limit has distributional derivative in $L^{4/3}(0,T;V')$; and finally, strong convergence in $L^2(0,T;H)$ implies weak convergence in $L^{4/3}(0,T;V')$. ■

Thus u lives in $L^2(0,T;V)$, $L^\infty(0,T;H)$, $W^{1,4/3}(0,T;V')$. The latter fact implies that (up to a representative of the equivalence class) u is continuous in V', as one can check from the identity

$$u(t) - u(t_0) = \int_{t_0}^t \frac{du(s)}{ds} ds,$$

a priori true for a.e. t_0 and t. Now, this implies that u is weakly continuous in H:

Lemma 39 *If u belongs to $L^{\infty}(0,T;H)$ and $C([0,T];V')$ then it is weakly continuous in H.*

Proof. Let $L = \sup_{[0,T]} |u|_H$. Given $\varphi \in H$, take a sequence $\{\varphi_n\} \subset V$, converging to φ in H. Let us prove the continuity of $\langle u(t), \varphi\rangle_H$ at point t_0. Given $\varepsilon > 0$, there is n such that $|\varphi - \varphi_n|_H \leq \frac{\varepsilon}{4L}$ and, corresponding to that n, there is $\delta > 0$ such that $|t - t_0| < \delta$, $t \in [0,T]$, implies $\left|\langle u(t) - u(t_0), \varphi_n\rangle_{V',V}\right| \leq \frac{\varepsilon}{2}$. Then, for such value of t,

$$\begin{aligned} &\langle u(t) - u(t_0), \varphi\rangle_H \\ &= \langle u(t) - u(t_0), \varphi - \varphi_n\rangle_H + \langle u(t) - u(t_0), \varphi_n\rangle_{V',V} \\ &\leq \varepsilon. \end{aligned}$$

The proof is complete. ■

Moreover, $u(0) = u_0$. Indeed, we know that for every $T' \in [0,T]$

$$\int_0^{T'} \left\langle \frac{du_{n_k}(t)}{dt}, \varphi(t) \right\rangle_{V',V} dt \to \int_0^{T'} \left\langle \frac{du(t)}{dt}, \varphi(t) \right\rangle_{V',V} dt$$

for every $\varphi \in L^4(0,T;V)$. But, for constant $\varphi \in V$, we may calculate the two integrals and get

$$\begin{aligned} &\langle u_{n_k}(T'), \varphi\rangle_{V',V} - \langle \pi_{n_k} u_0, \varphi\rangle_{V',V} \\ \to\ &\langle u(T'), \varphi\rangle_{V',V} - \langle u(0), \varphi\rangle_{V',V}. \end{aligned}$$

Choose now T' such that $u_{n_k}(T') \to u(T')$ in H (this is true for almost every T') and deduce that $\langle \pi_{n_k} u_0, \varphi\rangle_{V',V}$ converges to $\langle u(0), \varphi\rangle_{V',V}$. But we know that $\langle \pi_{n_k} u_0, \varphi\rangle_{V',V}$ converges to $\langle u_0, \varphi\rangle_{V',V}$, hence $u(0) = u_0$ by the arbitrariness of $\varphi \in V$.

Let us prove that u satisfies the Navier-Stokes equation in weak form. Let $\varphi : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth, divergence free, zero mean and compact support vector field. We know that the Galerkin system in differential is satisfied as an equality between $W^{1,2}(0,T;H_n)$ functions. Again we are going to use differential calculus in this class, but if the reader prefers to keep at the level of classical calculus, it is sufficient to approximate f_n further by continuous functions. We have

$$\begin{aligned} &\int_0^T \left\langle \frac{du_{n_k}(t)}{dt}, \varphi(t) \right\rangle_H dt + \int_0^T \langle A_{n_k} u_{n_k}(t), \varphi(t)\rangle_H dt \\ &= \int_0^T \langle f_{n_k}(t), \varphi(t)\rangle_H dt - \int_0^T \langle B_{n_k}(u_{n_k}(t), u_{n_k}(t)), \varphi(t)\rangle_H dt \end{aligned}$$

which easily yields

$$
\begin{aligned}
&-\int_0^T \left\langle u_{n_k}(t), \frac{d\varphi(t)}{dt} - A\varphi(t) \right\rangle_H dt \\
&= \int_0^T \langle f_{n_k}(t), \varphi(t) \rangle_{V',V} \, dt + \int_0^T \langle B_{n_k}(u_{n_k}(t), \pi_{n_k}\varphi(t)), u_{n_k}(t) \rangle_H \, dt.
\end{aligned}
$$

Strong convergence in $L^2(0,T;H)$ of $\{u_{n_k}\}$ to u allows to pass to the limit in the first term. The term with f_{n_k} is also easy.

Exercise 40 *Show that*

$$
\int_0^T \langle B_{n_k}(u_{n_k}(t), \pi_{n_k}\varphi(t)), u_{n_k}(t) \rangle_H \, dt
$$

converges to

$$
\int_0^T \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla\varphi) \, dxdt
$$

thanks to the strong convergence in $L^2(0,T;H)$ of $\{u_{n_k}\}$ to u.

The weak form of the equation for u is thus obtained. To avoid misunderstandings, let us stress the fact that strong convergence in $L^2(0,T;H)$ is a key point: just with weak convergence properties we could not pass to the limit in the quadratic term of the exercise.

Let us prove that the energy inequality is fulfilled. At the beginning of section 2.6.2 we have proved that

$$
|u_{n_k}(t)|_H^2 + \nu \int_{t_0}^t \|u_{n_k}(s)\|_V^2 \, ds = |u_{n_k}(t_0)|_H^2 + \int_{t_0}^t \langle f_{n_k}(s), u_{n_k}(s) \rangle_H \, ds.
$$

In fact we have proved it only for $t_0 = 0$, but there is no difference for general $t_0 \in [0,t]$. We know that there exists a zero-measure set $N \subset [0,T]$ such that $u_{n_k}(t) \to u(t)$ in H for every $t \in [0,T] \setminus N$. Since we have proved above that $u(0) = u_0$, it follows that $u_{n_k}(0) \to u(0)$ in H, hence we may assume that $0 \notin N$. Let $0 \le t_0 \le t \le T$ be two elements of $[0,T] \setminus N$. Then the terms $|u_{n_k}(t)|_H^2$ and $|u_{n_k}(t_0)|_H^2$ converge to $|u(t)|_H^2$ and $|u(t_0)|_H^2$ respectively. We know that $\{u_{n_k}\}$ converges weakly to u in $L^2(0,T;V)$ and it is easy to check that f_{n_k} strongly converges to f in $L^2(0,T;V')$; hence one easily verifies that

$\int_{t_0}^{t} \langle f_{n_k}(s), u_{n_k}(s)\rangle_H \, ds$ converges to $\int_{t_0}^{t} \langle f(s), u(s)\rangle_{V',V} \, ds$. Thus we deduce that $\nu \int_{t_0}^{t} \|u_{n_k}(s)\|_V^2 \, ds$ has a limit, call it $L(t_0, t)$, and

$$|u(t)|_H^2 + L(t_0, t) = |u(t_0)|_H^2 + \int_{t_0}^{t} \langle f(s), u(s)\rangle_{V',V} \, ds.$$

The point is that we cannot identify $L(t_0, t)$ with $\nu \int_{t_0}^{t} \|u(s)\|_V^2 \, ds$. From the weak convergence of $\{u_{n_k}\}$ to u in $L^2(0, T; V)$ we can only deduce

$$\int_{t_0}^{t} \|u(s)\|_V^2 \, ds \leq \liminf_{k\to\infty} \int_{t_0}^{t} \|u_{n_k}(s)\|_V^2 \, ds.$$

The latter is less or equal to $L(t_0, t)$. Hence we only get

$$|u(t)|_H^2 + \nu \int_{t_0}^{t} \|u(s)\|_V^2 \, ds \leq |u(t_0)|_H^2 + \int_{t_0}^{t} \langle f(s), u(s)\rangle_{V',V} \, ds.$$

This is the energy inequality in the form described in remark 21, that implies the energy inequality in the form required by the definition of Leray-Hopf weak solution.

2.7 A modified Navier-Stokes system

Motivated both by the problem of removing the zero-mean condition and by the generalization to stochastic problems, we may consider the following modified version of the Navier-Stokes system:

$$\begin{aligned} \frac{\partial u}{\partial t} + (u + b) \cdot \nabla u + \nabla p &= \nu \triangle u + f + V \cdot u \\ \mathtt{div} u &= 0 \\ u|_{t=0} &= u_0 \end{aligned}$$

where $b : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is a given measurable L-periodic divergence free field and $V : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^{3\times 3}$ is a matrix valued measurable L-periodic function. Assume for instance that

$$b \text{ satisfies (2.1)}, \qquad \sup_{(x,t)\in[0,L]^d\times[0,T]} |V(x, t)| < \infty$$

but less is sufficient on b and V to give the following definition and prove the theorem below (in all our examples b will be at least as regular as u itself, while V may depend).

Definition 41 *Given $u_0 \in H$, $f \in L^2(0,T;H)$, b and V as above, we say that a measurable vector field $u : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ is a periodic Leray-Hopf weak solution of the modified Navier-Stokes system above if*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V),$$

it is weakly continuous in H, $u|_{t=0} = u_0$, it satisfies the modified Navier-Stokes system in the weak form

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \left(\frac{\partial \varphi}{\partial t} + \triangle \varphi \right) dxdt + \int_0^T \int_{\mathbb{R}^3} u \cdot ((u+b) \cdot \nabla \varphi)\, dxdt = -\int_0^T \int_{\mathbb{R}^3} (f + V \cdot u) \cdot \varphi dxdt$$

for all $\varphi : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$, smooth, divergence free and compact support, and it satisfies the energy inequality

$$\frac{1}{2} \int_{[0,L]^d} |u(x,t)|^2 dx + \int_{t_0}^t \int_{[0,L]^d} |\nabla u|^2 dxds \leq \frac{1}{2} \int_{[0,L]^d} |u(x,t_0)|^2 dx + \int_{t_0}^t \int_{[0,L]^d} u \cdot (f + V \cdot u)\, dxds$$

for every $t > 0$ and a.e. $t_0 \in [0,t]$, including $t_0 = 0$.

As usual, we can write a variant for $f \in L^2(0,T;V')$ using the dual pairing $\langle f(t), u(.,t) \rangle_{V',V}$ in place of $\int_{[0,L]^d} u \cdot f dx$.

Theorem 42 *There exists at least one periodic Leray-Hopf weak solution of the modified Navier-Stokes system.*

The proof under the assumptions above for b and V is similar to the case treated before and thus is a useful exercise.

2.8 Appendix A

2.8.1 Derivatives in the sense of distributions

We denote by $C_0^\infty(\mathbb{R}^d)$ the space of all function $\varphi : \mathbb{R}^d \to \mathbb{R}$, which are infinitely differentiable and have compact support.

We say that a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is locally integrable if $\int_B |f(x)|\, dx < \infty$ for all bounded Borel sets $B \subset \mathbb{R}^d$.

Given a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $\partial_k f$ the linear mapping from $C_0^\infty\left(\mathbb{R}^d\right)$ to $\mathbb{R}$ defined as

$$\partial_k f(\varphi) := -\int_{\mathbb{R}^d} f(x)\, \partial_k \varphi(x)\, dx, \quad \varphi \in C_0^\infty\left(\mathbb{R}^d\right).$$

This definition is clearly motivated by the fact that, if f is differentiable and $\partial_k f$ is locally integrable, then the "integral action" of $\partial_k f$ on $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$ defined as $\int_{\mathbb{R}^d} \partial_k f(x)\, \varphi(x)\, dx$ is given by $-\int_{\mathbb{R}^d} f(x)\, \partial_k \varphi(x)\, dx$, by Gauss-Green theorem.

For all kind of differential operators we use similar convention. For instance, the divergence in the sense of distributions of a locally integrable vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ is the linear mapping from $C_0^\infty\left(\mathbb{R}^d\right)$ to $\mathbb{R}$ defined as

$$\mathtt{div} v(\varphi) := -\int_{\mathbb{R}^d} v(x) \cdot \nabla \varphi(x)\, dx, \quad \varphi \in C_0^\infty\left(\mathbb{R}^d\right).$$

Given a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$, we say that $\partial_k f$ (in the sense of distributions) has a property $\mathcal{P}$, if there is a locally integrable function $g : \mathbb{R}^d \to \mathbb{R}$ having property $\mathcal{P}$ such that

$$\partial_k f(\varphi) := \int_{\mathbb{R}^d} g(x)\, \varphi(x)\, dx, \quad \varphi \in C_0^\infty\left(\mathbb{R}^d\right).$$

This is the way, in the appendix to Chapter 1, we have defined the spaces $W^{1,p}$.

2.8.2 Boundary conditions for divergence free periodic fields

Preliminarily, we recall without details a fact on traces on the boundary for functions of class $W^{1,p}$. First, one should not forget that functions in spaces like L^p and $W^{1,p}$ are classes of equivalence: two functions which are equal almost surely are considered the same element of the space, otherwise we would get contradictions like that a Cauchy sequence in L^p converges to infinitely many different functions. Standing this convention, for a generic function $f \in L^p(D)$, on an open set $D \subset \mathbb{R}^3$, the value of f at a point, or

along a line, or on a surface is not a well defined concept: $f|_A$ is meaningless if A is a Lebesgue measure zero set. But if D is a Lipschitz domain and $f \in W^{1,p}(D)$, then $f|_{\partial D}$ is a well defined $L^p(\partial D)$-function, see **??**. The point is that a surface has codimension one. In a sense the previous result corresponds to the fact that in one space dimension $W^{1,p}$ is embedded into the space of continuous functions, hence the restriction to single points is well defined.

If a vector field v is of class $H^1_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$, in particular it is of class $W^{1,2}\left([0,L]^3\right)$, hence $v|_{\partial[0,L]^3}$ is well defined. The same for $v|_{\partial\left([0,L]^3+\underline{k}L\right)}$ for every $\underline{k}\in\mathbb{Z}^3$. But the equality $v = v(\cdot+\underline{k}L)$ implies that $v|_{\partial\left([0,L]^3+\underline{k}L\right)}$ is independent of $\underline{k}$. In this sense we speak of *periodic boundary conditions.*

If v is only of class $L^2_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$, a priori we cannot define $v|_{\partial[0,L]^3}$. However, due to the property $\mathtt{div}v = 0$, we can define in a natural way at least the normal trace $v\cdot n|_{\partial[0,L]^3}$. But the object we obtain is a distribution on $\partial[0,L]^3$. Let us give a few details.

Consider first S to be the surface $x_1 = 0$ and say that n on S is the vector $(1,0,0)$. For every $v\in C^\infty_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$, the function

$$v\cdot n|_S = v_1(0,.,.)$$

is obviously well defined. We have a linear mapping

$$C^\infty_{per,div}\left(\mathbb{R}^3,\mathbb{R}^3\right)\ni v\mapsto v\cdot n|_S\in C^\infty(S).$$

Let us consider $v\cdot n|_S$ as the following distribution on S:

$$v\cdot n|_S(\varphi) := \int_S v_1(0,\sigma)\,\varphi(\sigma)\,d\sigma,\quad \varphi\in C^\infty_0(S).$$

It is convenient to localize and thus, given $R>0$, denoted by S_R the open ball in S of radius R and center in the origin, we consider also the distribution

$$v\cdot n|_{S_R}(\varphi) := \int_{S_R} v_1(0,\sigma)\,\varphi(\sigma)\,d\sigma,\quad \varphi\in C^\infty_0(S_R).$$

Lemma 43 *There is a constant $C_R>0$ such that*

$$|v\cdot n|_{S_R}(\varphi)|\le C_R\,|v|_H\,\|\varphi\|_{W^{1,2}(S_R)}$$

for every $v\in C^\infty_{per,div}(\mathbb{R}^3,\mathbb{R}^3)$ and $\varphi\in C^\infty_0(S_R)$.

Proof. Let D_R be the cylinder $x_1 \in [0, L]$, $(x_2, x_3) \in S_R$. Given $\varphi \in C_0^\infty(S_R)$, extend φ to a function $\widetilde{\varphi}$ on D_R in the trivial way by setting $\widetilde{\varphi}(x_1, x_2, x_3) = \varphi(x_2, x_3)$. Take a smooth, non negative function $\theta(x_1)$, $x_1 \in [0, L]$, equal to one for $x_1 \in \left[0, \frac{L}{3}\right]$ and equal to zero for $x_1 \in \left[\frac{2L}{3}, L\right]$. Then define the new function $\varphi^*(x_1, x_2, x_3) = \widetilde{\varphi}(x_1, x_2, x_3)\,\theta(x_1)$.

We have

$$\int_{D_R} v \cdot \nabla \varphi^* dx = -\int_{D_R} \varphi^* \mathtt{div} v dx + \int_{S_R} v \cdot n \varphi^* d\sigma$$
$$= \int_{S_R} v \cdot n \varphi d\sigma$$

because φ^* is zero on $\partial D_R \setminus S_R$ and $\mathtt{div} v = 0$. Hence

$$\left| \int_{S_R} v \cdot n \varphi d\sigma \right| \leq \left(\int_{D_R} |v|^2 dx \right)^{1/2} \left(\int_{D_R} |\nabla \varphi^*|^2 dx \right)^{1/2}.$$

By easy calculations this implies the result. The proof is complete. ■

Recall that $C^\infty_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ is dense in $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ (see section 2.2). Hence, given $v \in L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ we define

$$v \cdot n|_{S_R}(\varphi) := \lim_{\varepsilon \to 0} v_\varepsilon \cdot n|_{S_R}(\varphi), \quad \varphi \in C_0^\infty(S_R)$$

where $\{v_\varepsilon\} \subset C^\infty_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ converges to v in $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$, being sure that the limit exists and is independent of $\{v_\varepsilon\}$ thanks to the lemma.

We omit the easy proof of the following fact: if $R_1 > R_2$, $v \cdot n|_{S_{R_1}}(\varphi) = v \cdot n|_{S_{R_2}}(\varphi)$ for all $\varphi \in C_0^\infty(S_{R_2})$. This implies that we can define $v \cdot n|_S(\varphi)$, for all $v \in L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$ and $\varphi \in C_0^\infty(S)$, by setting

$$v \cdot n|_S(\varphi) = v \cdot n|_{S_R}(\varphi)$$

where R is any radius such that S_R is larger than the support of φ: no ambiguity may occur.

Obviously we can define $v \cdot n|_{S'}(\varphi)$ over the surface S' defined by $x_1 = L$. By the construction it follows that

$$v \cdot n|_S(\varphi) = v \cdot n|_{S'}(\varphi).$$

This is the periodicity on the boundary that we wanted to prove, for fields v of class $L^2_{per,div}(\mathbb{R}^3, \mathbb{R}^3)$. Of course it holds in every directions.

2.8.3 Dual spaces

We mention just a few facts on dual spaces in the Hilbertian case. If H is a Hilbert space, its *dual* is the space H' is the space of all *functional* h' on H, namely all bounded linear mappings from H to $\mathbb{R}$. We call *dual pairing* the bilinear form defined on $H' \times H$ by $(h', h) \mapsto h'(h)$ and we denote it also by $\langle h', h \rangle_{H',H}$. Namely $\langle h', h \rangle_{H',H} = h'(h)$ for all $h' \in H'$, $h \in H$. This definition makes sense for Banach spaces too and also for more general structures.

If H is a Hilbert space with inner product $\langle ., . \rangle_H$, any element $h \in H$ defines a functional $h' \in H'$, simply by $h'(\varphi) = \langle h, \varphi \rangle_H$, $\varphi \in H$. Viceversa, by Riesz theorem, if $h' \in H'$, there exists $h \in H$, unique, such that $h'(\varphi) = \langle h, \varphi \rangle_H$, $\varphi \in H$. Denote by $i(h')$ this unique element h; the mapping $i : H' \to H$ defined by Riesz theorem is an isomorphism. We thus have, by definition,

$$h'(\varphi) = \langle i(h'), \varphi \rangle_H$$

for all $h' \in H'$, $\varphi \in H$.

Let $V \subset H$ be another Hilbert space, continuously embedded into H. Let V' be its dual space and $\langle v', v \rangle_{V',V}$ be the dual pairing. The space H' is naturally embedded into V', because given $h' \in H'$ we can define the element $\iota_{H',V'}(h') \in V'$ simply by setting $\iota_{H',V'}(h')(v) = h'(v)$ for all $v \in V$. In other words, $\iota_{H',V'}(h')$ is the restriction of h' to V. We have defined a bounded linear mapping $\iota_{H',V'} : H' \to V'$. Since there is usually no possibility of misunderstandings, we denote $\iota_{H',V'}(h')$ simply by h', namely we say that if $h' \in H'$ then h' is also an element of V', and finally we also write $H' \subset V'$ understanding the inclusion in the previous sense.

We can go further and interpret H as a subspace of V'. Given $h \in H$, we associate to it first $h' = i(h) \in H'$, then

$$\iota_{H',V'}(h') = (\iota_{H',V'} \circ i)(h) \in V'.$$

When we loosely write $h \in V'$ we mean the previous sentence. In other words, when we say $h \in V'$ we mean that h is the functional on V defined as

$$h(v) = \langle h, v \rangle_H .$$

This very simple vision coincides with the more complicate above since

$$(\iota_{H',V'} \circ i)(h)(v) = \iota_{H',V'}(i(h))(v) = i(h)(v) = \langle h, v \rangle_H .$$

In terms of dual pairing we also have the important identity

$$\langle h, v\rangle_{V',V} = \langle h, v\rangle_H \text{ for all } h \in H, v \in V.$$

Here, in the notation $\langle h, v\rangle_{V',V}$, we understand that $h \in V'$ means $(\iota_{H',V'} \circ i)(h) \in V'$ as just explained.

2.9 Appendix B

2.9.1 Fourier analysis of periodic L^2 functions

Given $L > 0$, denote by $L^2_{per}\left(\mathbb{R}^d; \mathbb{C}\right)$ the space of all L-periodic measurable functions $v : \mathbb{R}^d \to \mathbb{C}$, square integrable on $[0, L]^d$, with inner product

$$\langle u, v\rangle = \int_{[0,L]^d} u(x')\,\overline{v(x')}dx'$$

It is a complex Hilbert space. It is obviously isomorphic to the space $L^2\left([0, L]^d; \mathbb{C}\right)$ of all square integrable functions $v : [0, L]^d \to \mathbb{C}$, endowed with the same inner product. For homogeneity with the notations used in the Chapter, we prefer to see them as periodic functions defined over the full space.

The family of functions $\left\{e^{\frac{2\pi}{L}ik\cdot x}\right\}_{k\in\mathbb{Z}^d}$ is a complete orthonormal system. For every $v \in L^2_{per}\left(\mathbb{R}^d; \mathbb{C}\right)$ we have

$$v(x) = \sum_{k\in\mathbb{Z}^d} v^{(k)} e^{\frac{2\pi i}{L}k\cdot x}, \quad v^{(k)} := \int_{[0,L]^d} v(x')\, e^{\frac{2\pi i}{L}k\cdot x'}dx'. \tag{2.4}$$

We also have the Parseval relation

$$\int_{[0,L]^d} |v(x)|^2\, dx = \sum_{k\in\mathbb{Z}^d} \left|v^{(k)}\right|^2.$$

The space $L^2_{per}\left(\mathbb{R}^d; \mathbb{R}\right)$ (real valued) is a closed subspace of $L^2_{per}\left(\mathbb{R}^d; \mathbb{C}\right)$.

Exercise 44 *$v \in L^2_{per}\left(\mathbb{R}^d; \mathbb{R}\right)$ if and only if $v^{(-k)} = \overline{v^{(k)}}$ for every $k \in \mathbb{Z}^d$.*

Hence

$$\begin{aligned} v(x) &= \frac{1}{2}\sum_{k\in\mathbb{Z}^d} v^{(k)} e^{\frac{2\pi i}{L}k\cdot x} + \frac{1}{2}\sum_{k\in\mathbb{Z}^d} v^{(-k)} e^{-\frac{2\pi i}{L}k\cdot x} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}^d} v^{(k)}\left[\cos\left(\frac{2\pi}{L}k\cdot x\right) + i\sin\left(\frac{2\pi}{L}k\cdot x\right)\right] \\ &+ \frac{1}{2}\sum_{k\in\mathbb{Z}^d} \overline{v^{(k)}}\left[\cos\left(\frac{2\pi}{L}k\cdot x\right) - i\sin\left(\frac{2\pi}{L}k\cdot x\right)\right] \\ &= \sum_{k\in\mathbb{Z}^d} \operatorname{Re} v^{(k)} \cos\left(\frac{2\pi}{L}k\cdot x\right) - \sum_{k\in\mathbb{Z}^d} \operatorname{Im} v^{(k)} \sin\left(\frac{2\pi}{L}k\cdot x\right). \end{aligned}$$

If we define

$$v^{(k,\cos)} := \int_{T^d} v\left(x'\right)\cos\left(\frac{2\pi}{L}k\cdot x'\right)dx', \quad v^{(k,\sin)} := \int_{T^d} v\left(x'\right)\sin\left(\frac{2\pi}{L}k\cdot x'\right)dx' \tag{2.5}$$

we get

$$v(x) = \sum_{k\in\mathbb{Z}^d}\left(v^{(k,\cos)}\cos\left(\frac{2\pi}{L}k\cdot x\right) - v^{(k,\sin)}\sin\left(\frac{2\pi}{L}k\cdot x\right)\right). \tag{2.6}$$

Thus a complete orthonormal system of $L^2_{per}\left(\mathbb{R}^d;\mathbb{R}\right)$ is

$$\left\{\cos\left(\frac{2\pi}{L}k\cdot x\right), \sin\left(\frac{2\pi}{L}k\cdot x\right)\right\}_{k\in\mathbb{Z}^d}.$$

2.9.2 Sobolev spaces

Denote by $l^2_{\mathbb{C}}$ the space of all families $\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d} \subset \mathbb{C}$ such that $\sum_{k\in\mathbb{Z}^d}\left|v^{(k)}\right|^2 < \infty$. The mapping $v \mapsto \left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}$ (see (2.4)) defines a mapping $\mathcal{F}$ from $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ to $l^2_{\mathbb{C}}$, which is easily proved to be an isometry. For every $\alpha \geq 0$, let us define $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ as the space of all $v \in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ such that

$$\left\|\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}\right\|^2_\alpha := \sum_{k\in\mathbb{Z}^d}\left(1+\|k\|^2\right)^{\alpha/2}\left|v^{(k)}\right|^2 < \infty.$$

For the interpretation in terms of derivatives, one can verify, by means of (2.4)), that $H^n_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$, $n \in \mathbb{N}$, are the usual Sobolev spaces of (complex

valued) functions with square integrable derivatives up to order n. This way one can check that there is no difference between that space $H^1_{per,div}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ defined at the end of this appendix and the space used in Chapter 2.

One could identify $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ with the space of families $\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}$ such that $\left\|\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}\right\|^2_\alpha < \infty$. From this viewpoint it is natural to introduce also the space $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ for $\alpha < 0$, again as the space of families $\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}$ such that $\left\|\left\{v^{(k)}\right\}_{k\in\mathbb{Z}^d}\right\|^2_\alpha < \infty$. This way we have as sort of general definition of $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ for every $\alpha\in\mathbb{R}$.

In appendix A we have defined the derivative in the sense of distributions of an L^2-function. We may use these negative-order Sobolev spaces to accommodate them.

Exercise 45 *Given $v\in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$, prove that $\frac{\partial v}{\partial x_j}\in H^{-1}_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$, under proper natural identifications.*

Proof. Hint: Show that, for every test function φ,

$$\left(\frac{\partial v}{\partial x_j}\right)(\varphi) = \sum_{k\in\mathbb{Z}^d}(\varphi)^{(k)}\,\frac{2\pi i}{L}k_j v^{(k)}.$$

■

We could write

$$v\in L^2\left(T^d;\mathbb{C}\right)\Rightarrow \frac{\partial v}{\partial x_j} = \left\{\frac{2\pi i}{L}k_j v^{(k)}\right\}_{k\in\mathbb{Z}^d}\in H^{-1}\left(T^d;\mathbb{C}\right). \tag{2.7}$$

Finally, let us discuss the real-valued case. One one side we could say that $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{R}\right)$ is defined by the two conditions

$$\begin{cases} \sum_{k\in\mathbb{Z}^d}\left(1+\|k\|^2\right)^{\alpha/2}\left|v^{(k)}\right|^2<\infty \\ v^{(-k)}=\overline{v^{(k)}} \text{ for every } k\in\mathbb{Z}^d. \end{cases}$$

On the other side, if we want to refer to the Fourier analysis (2.5)-(2.6), we could say that $H^\alpha_{per}\left(\mathbb{R}^d;\mathbb{R}\right)$ is defined by the condition

$$\sum_{k\in\mathbb{Z}^d}\left(1+\|k\|^2\right)^{\alpha/2}\left(\left|v^{(k,\cos)}\right|^2+\left|v^{(k,\sin)}\right|^2\right)<\infty.$$

We have the following important result (a variant of Rellich theorem).

Theorem 46 *$H^{\alpha}_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ is compactly embedded into $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$, for every $\alpha>0$. The same is true for $H^{\alpha}_{per}\left(\mathbb{R}^d;\mathbb{R}\right)$ and $L^2_{per}\left(\mathbb{R}^d;\mathbb{R}\right)$.*

Proof. Consider a sequence $\{v_n\}_{n\in\mathbb{N}}\subset H^{\alpha}_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ such that

$$\sum_{k\in\mathbb{Z}^d}\left(1+\|k\|^2\right)^{\alpha/2}\left|v_n^{(k)}\right|^2\leq C$$

for some constant $C>0$. We want to prove that there is a subsequence $\{v_{n'}\}_{n\in\mathbb{N}}$ which converges to some $v\in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$.

It is clear that for every k one can extract a subsequence of $\left\{v_n^{(k)}\right\}_{n\in\mathbb{N}}$ which converges to some complex number. By a diagonal procedure, there is a subsequence $\{v_{n'}\}$ and some $v\in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$ such that

$$v_{n'}^{(k)}\to v^{(k)}\quad\text{ for every }k.$$

We have

$$\sum_{|k|\geq R}\left|v_n^{(k)}\right|^2=\sum_{|k|\geq R}\frac{\left(1+\|k\|^2\right)^{\alpha/2}}{\left(1+\|k\|^2\right)^{\alpha/2}}\left|v_n^{(k)}\right|^2\leq\frac{C}{(1+R^2)^{\alpha/2}}$$

hence, given $\varepsilon>0$, there is $R>0$ such that

$$\sum_{|k|>R}\left|v_n^{(k)}\right|^2\leq\frac{\varepsilon}{4}\quad\text{ for every }n.$$

This is the key fact. Together with the property of $\{v_{n'}\}$, it yields

$$\sum_{|k|>R}\left|v^{(k)}\right|^2\leq\frac{\varepsilon}{4}$$

$$\sum_{k\in\mathbb{Z}^d}\left|v_{n'}^{(k)}-v^{(k)}\right|^2\leq\sum_{|k|\leq R}\left|v_{n'}^{(k)}-v^{(k)}\right|^2+\varepsilon$$

hence $\{v_{n'}\}$ converges to v in $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$.

The proof in the real valued case is similar or a consequence of the closedness of the real-valued spaces in the complex-valued ones. The proof is complete. ■

2.9.3 Fourier analysis of divergence free periodic L^2 vector fields

Periodic vector fields

Let $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$ be the space of all L-periodic measurable vector fields $v : \mathbb{R}^d \to \mathbb{C}^d$ with square integrable components on $[0,L]^d$. The components $v_l : \mathbb{R}^d \to \mathbb{C}$, $l = 1,...,d$, belong to $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}\right)$. The space $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$, with the usual inner product

$$\langle u, v\rangle = \int_{T^d} u\left(x'\right)\cdot\overline{v\left(x'\right)}dx'$$

is an Hilbert space. A complete orthonormal system of $L^2_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$ is given by the fields

$$h^{(k,j)}(x) := a^{(k,j)}e^{\frac{2\pi i}{L}k\cdot x}, \quad k\in\mathbb{Z}^d, j = 1,...,d$$

where $a^{(k,1)},...,a^{(k,d)}$ is, for each k, any orthonormal basis of $\mathbb{C}^d$ (see the exercise below). We can write two interesting Fourier-type developments, in the vector valued case. Given $v\in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$, define $v_a^{(k)}, v^{(k)}\in\mathbb{C}^d$, $k\in\mathbb{Z}^d$, as the vectors with components

$$v_a^{(k,j)} := \int_{[0,L]^d} v\cdot h^{(k,j)}dx', \quad v_j^{(k)} := \int_{[0,L]^d} v_j\left(x'\right)e^{\frac{2\pi i}{L}k\cdot x'}dx' \tag{2.8}$$

(notice that $v_a^{(k,j)} = v^{(k)}\cdot a^{(k,j)}$) for $j = 1,...,d$; then

$$v\left(x\right) = \sum_{\substack{k\in\mathbb{Z}^d\\ j=1,..,d}} v_a^{(k,j)}h^{(k,j)}(x) \quad \text{and also} \quad v\left(x\right) = \sum_{k\in\mathbb{Z}^d} v^{(k)}e^{\frac{2\pi i}{L}k\cdot x}. \tag{2.9}$$

Moreover (Parseval relations)

$$\int_{[0,L]^d}\left|v\left(x\right)\right|^2 dx = \sum_{\substack{k\in\mathbb{Z}^d\\ j=1,..,d}}\left|v_j^{(k)}\right|^2 = \sum_{k\in\mathbb{Z}^d}\left|v^{(k)}\right|^2.$$

Exercise 47 *Prove (2.9) starting from the results of the previous section on periodic functions.*

Proof. Hint: Notice that

$$\sum_k \left(v\left(x\right)\cdot a^{(k,j)}\right)a^{(k,j)} = v\left(x\right).$$

■

The (natural) definition of $H^{\alpha}_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$ is obvious and left to the reader.

Let us discuss the case with real-valued components. If $v \in L^2_{per}\left(\mathbb{R}^d;\mathbb{R}^d\right)$, then arguing componentwise from the second development in (2.9) we get

$$v(x) = \sum_{k\in\mathbb{Z}^d}\left(v^{(k,\cos)}\cos\left(\frac{2\pi}{L}k\cdot x\right) + v^{(k,\sin)}\sin\left(\frac{2\pi}{L}k\cdot x\right)\right)$$

where now $v^{(k,\cos)}$ and $v^{(k,\sin)}$ are d-dim. real vectors, i.e. elements of $\mathbb{R}^d$. A complete orthonormal system of $L^2_{per}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ is thus

$$\left\{a^{(k,j)}\cos\left(\frac{2\pi}{L}k\cdot x\right), b^{(k,j)}\sin\left(\frac{2\pi}{L}k\cdot x\right)\right\}_{k\in\mathbb{Z}^d,\alpha=1,...,d}$$

where $a^{(k,j)}$, $b^{(k,j)}$, $j=1,...,d$, is any pair of orthonormal basis of $\mathbb{R}^d$ (possibly depending on k).

In terms of the complex-valued developments (2.9), $v \in L^2_{per}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ if and only if

$$v_j^{(-k)} = \overline{v_j^{(k)}} \quad \text{for all } k\in\mathbb{Z}^d \text{ and } j=1,...,d.$$

If the basis $a^{(k,1)},...,a^{(k,d)}$ of $\mathbb{C}^d$ chosen in the complex-valued case satisfies $a^{(-k,j)} = \overline{a^{(k,j)}}$, then $v \in L^2_{per}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ if and only if

$$v_a^{(-k,j)} = \overline{v_a^{(k,j)}} \quad \text{for all } k\in\mathbb{Z}^d \text{ and } j=1,...,d.$$

Divergence free periodic fields

Let $L^2_{per,div}\left(\mathbb{R}^d;\mathbb{C}^d\right)$ be the space of all $v \in L^2_{per}\left(\mathbb{R}^d;\mathbb{C}^d\right)$ with $\mathtt{div}v = 0$ in the sense of distributions on $\mathbb{R}^d$, as explained in appendix 1 and the beginning of Chapter 2.

Since

$$\left(\mathtt{div}v\right)\left(\varphi\right) = -\int_{\mathbb{R}^d}\nabla\varphi\left(x\right)\cdot v\left(x\right)dx, \quad \varphi\in C_0^{\infty}\left(\mathbb{R}^d;\mathbb{C}\right),$$

using the identification (2.7) above, we may also write

$$\mathtt{div} v = \left\{ \frac{2\pi i}{L} k \cdot v^{(k)} \right\}_{k \in \mathbb{Z}^d} \in H^{-1}_{per}\left(\mathbb{R}^d; \mathbb{C}\right).$$

The real Hilbert space $L^2_{per,div}\left(\mathbb{R}^d; \mathbb{R}^d\right)$ is similarly defined.

The basis elements $h^{(k,j)}(x) = a^{(k,j)} e^{\frac{2\pi i}{L} k \cdot x}$ belong to $L^2_{per,div}\left(\mathbb{R}^d; \mathbb{C}^d\right)$ if and only if

$$k \cdot a^{(k,j)} = 0.$$

It is then clear that a complete orthonormal system of $L^2_{per,div}\left(\mathbb{R}^d; \mathbb{C}^d\right)$ is given by the fields

$$a^{(k,j)} e^{\frac{2\pi i}{L} k \cdot x}, \quad k \in \mathbb{Z}^d, j = 1, ..., d-1$$

where $a^{(k,1)}, ..., a^{(k,d-1)}$ is, for each k, any orthonormal basis of $\Lambda^{\mathbb{C}}_k$, the $(d-1)$-dimensional orthogonal space to k in $\mathbb{C}^d$.

Similarly, a complete orthonormal system of $L^2_{per,div}\left(\mathbb{R}^d; \mathbb{R}^d\right)$ is

$$\left\{ a^{(k,j)} \cos\left(\frac{2\pi}{L} k \cdot x\right), b^{(k,j)} \sin\left(\frac{2\pi}{L} k \cdot x\right) \right\}_{k \in \mathbb{Z}^d, \alpha = 1, ..., d-1}$$

where $a^{(k,j)}$, $b^{(k,j)}$, $j = 1, ..., d-1$, is any pair of orthonormal basis of Λ_k, the $(d-1)$-dimensional orthogonal space to k in $\mathbb{R}^d$.

2.9.4 Examples of fields such that $u \cdot \nabla u = 0$

Consider sufficiently regular divergence free periodic vector fields $u : \mathbb{R}^3 \to \mathbb{R}^3$. We would like to understand the condition

$$u \cdot \nabla u = 0.$$

It is not clear how to characterize these fields. If we find a solution, it is a constant-in-time solution of Euler equation, with associated pressure equal to a constant.

We describe an easy particular case.

Proposition 48 *Given $a \in \mathbb{R}^3$, $k \in \mathbb{Z}^d$ and a differentiable L-periodic function $\theta : \mathbb{R} \to \mathbb{R}$, define the vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$*

$$u(x) = a\theta(k \cdot x).$$

If

$$a \cdot k = 0$$

then $u \in L^2_{per,div}(\mathbb{R}^3;\mathbb{R}^3)$ *and satisfies* $u \cdot \nabla u = 0$.

The proof is an easy exercise. The basis elements

$$\left\{ a^{(k,j)} \cos\left(\frac{2\pi}{L} k \cdot x\right), b^{(k,j)} \sin\left(\frac{2\pi}{L} k \cdot x\right) \right\}_{k \in \mathbb{Z}^d, \alpha = 1,\ldots,d-1}$$

with $a^{(k,j)}, b^{(k,j)} \in \Lambda_k$ of the previous section are examples of such fields.

Exercise 49 *Assume a vector field* $u : \mathbb{R}^3 \to \mathbb{R}^3$ *has the form* $u(x) = a\theta(k \cdot x)$ *for some* $a, k \in \mathbb{R}^3$ *and it satisfies* $\mathtt{div} u = 0$. *Prove that either* $a \cdot k = 0$ *or* θ *is constant; in any case* $u \cdot \nabla u = 0$.

Proposition 50 *If, in addition to the assumptions of the previous proposition,*

$$\theta'' = -\theta$$

then the function $u(t,x) = e^{-\nu \|k\|^2 t} a\theta(k \cdot x)$ *is an* L*-periodic solution of the Navier-Stokes equations.*

Also this proof is an easy exercise. of the Stokes operator. Solutions of the Navier-Stokes equations are for instance

$$e^{-\nu \|k\|^2 t} a^{(k,j)} \sin\left(\frac{2\pi}{L} k \cdot x\right), \quad e^{-\nu \|k\|^2 t} b^{(k,j)} \cos\left(\frac{2\pi}{L} k \cdot x\right).$$

Heuristic dynamical system considerations

Let us use in a rather heuristic way the notations of remark 32. The following geometry of the infinite-dimensional vector field $B(\cdot,\cdot)$ in H emerges. It points orthogonal to u: $\langle B(u,u), u \rangle = 0$; thus it rotates around the origin. But it is zero along the axes associated to the complete orthonormal system $\{h_n\}$. So it is certainly not a simple rotation. It is more a complex motion on spheres centered at the origin. Moreover,

$$\mathtt{div} B(u,u) = 0$$

in the sense that

$$\sum_k \frac{\partial \langle B(u,u), h_k\rangle}{\partial u_k} = \sum_k \sum_{k'} \sum_{k''} \langle B(h_{k'}, h_{k''}), h_k\rangle \frac{\partial u_{k'} u_{k''}}{\partial u_k}$$
$$= \sum_k \sum_{k' \neq k''} \langle B(h_{k'}, h_{k''}), h_k\rangle \frac{\partial u_{k'} u_{k''}}{\partial u_k}$$

because $B(h_{k'}, h_{k'}) = 0$,

$$= \sum_k \sum_{k \neq k''} \langle B(h_k, h_{k''}), h_k\rangle u_{k''} + \sum_k \sum_{k' \neq k} \langle B(h_{k'}, h_k), h_k\rangle u_{k'}$$
$$= \sum_k \sum_{k \neq k''} \langle B(h_k, h_{k''}), h_k\rangle u_{k''}$$
$$= -\sum_k \sum_{k \neq k''} \langle B(h_{k''}, h_k), h_k\rangle u_{k''} = 0.$$

So the motion associated to B alone is rotational, in some sense. The idea is that it rotates at least around each one of the main axes.

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